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CORRIGENDUM

Volume **231**, Number 2 (1999), in the article "Iterated Function System and Ruelle Operator," by Ai Hua Fan and Ka-Sing Lau, pages 319–344 (doi:10.1006/jmaa.1998.6210):

1. INTRODUCTION

We adopt the same notation as in [FL]. Let $\{w_j\}_{j=1}^N$ be a finite family of contractive, one-to-one self-conformal maps on an open set $V \subseteq \mathbb{R}^d$ with

$$0 < \inf_{x,j} |w_j'(x)| \leq \sup_{x,j} |w_j'(x)| < 1 \tag{1.1}$$

and all the $|w_j'|$ satisfying the Dini condition. Let K be the invariant set under $\{w_j\}_{j=1}^N$; i.e., $K = \cup_{j=1}^N w_j(K)$. We say that $\{w_j\}_{j=1}^N$ satisfies the *open set condition* (OSC) if there exists a bounded open set $U \subseteq V$ such that

$$w_j(U) \subseteq U \quad \text{and} \quad w_i(U) \cap w_j(U) = \emptyset \quad \text{for } i \neq j,$$

and the *strong open set condition* (SOSC) if in addition, the above bounded open set U can be chosen so that $U \cap K \neq \emptyset$. The SOSC has technical importance [FL]. Schief [S] proved, among the other results, that the OSC implies the SOSC for self-similar maps. In [FL, Lemma 2.6] we claimed the result for the self-conformal maps. However, it was pointed out by Peres *et al* [P] (and also by Patzschke and Öberg) that there is a gap in the proof and they also provided a new proof. Their proof involves a delicate extension of Schief's method and seems to be quite complicated. Here we give a much simpler argument to close up the gap (Theorem 3.3). It involves some strategic change of Schief's construction. We include some details here so that it can be read independently.

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2. THE CONSTRUCTION

We let $\mathcal{J}$ denote the set of finite indices $J = j_1 \cdots j_n,\ 1 \le j_i \le N$, and let

$$w_J = w_{j_1} \circ \cdots \circ w_{j_n}, \quad K_J = w_J(K), \quad r_J = \text{diam } K_J \ .$$

For convenience we assume that V is connected so that we can use the mean value theorem freely in Lemma 2.1. The condition is not essential and can be omitted, as is proved in [Y]. We have

LEMMA 2.1. *For the IFS* $\{w_j\}_{j=1}^N$,

(*i*) *there exists* $c_1 > 0$ *such that for any* $x, y \in V,\ J \in \mathcal{J}$,

$$c_1^{-1} r_J \le \frac{|w_J(x) - w_J(y)|}{|x - y|} \le c_1 r_J \ ; \tag{2.1}$$

(*ii*) *there exists* $c_2 > 1$ *such that for any* $I, J \in \mathcal{J}$,

$$c_2^{-1} r_I r_J \le r_{IJ} \le c_2 r_I r_J \ . \tag{2.2}$$

Proof. See [FL, Lemma 2.3 and (2.4)] for an elementary proof. Note that in [FL], the notation r_J is $|w_J'(x_0)|$ for some fixed x_0 in V; it differs from the r_J here by a universal constant. ∎

For any fixed $\varepsilon > 0$ and for any set $A \subseteq \mathbb{R}^d$, we let $B(A, \varepsilon) = \{y \in V : d(x, y) < \varepsilon$ for some $x \in A\}$; $B(x, \varepsilon)$ is the ε-ball in V center at x. Let

$$G_J = w_J(B(K, \varepsilon)).$$

By Lemma 2.1(i), we have, for any $x \in V$,

$$B(w_J(x), c_1^{-1}\varepsilon r_J) \subseteq w_J(B(x, \varepsilon)) \subseteq B(w_J(x), c_1 \varepsilon r_J). \tag{2.3}$$

It follows that

$$B(K_J, c_1^{-1}\varepsilon r_J) \subseteq G_J \subseteq B(K_J, c_1 \varepsilon r_J). \tag{2.4}$$

For $0 < b < 1$, we let

$$\Lambda_b = \{J = j_1 \cdots j_n : \quad r_{j_1 \cdots j_n} < b \le r_{j_1 \cdots j_{n-1}}\}.$$

Our most crucial difference from [S, P] is the following inductive way of defining the index set $\Lambda(J), J \in \mathcal{J}$: For $J = j$, we define

$$\Lambda(J) = \{I \in \Lambda_{\text{diam} G_J} : \ K_I \cap G_J \ne \varnothing\}.$$

Suppose $\Lambda(J)$ is defined; we define

$$\Lambda(jJ) = \mathscr{A} \cup \mathscr{B},$$

where

$$\mathscr{A} = \{jI : I \in \Lambda(J)\}$$

and

$$\mathscr{B} = \{I \in \Lambda_{\mathrm{diamG_{jJ}}} :\ i_1 \neq j \text{ and } K_I \cap G_{jJ} \neq \emptyset\}.$$

(Note that in [S], $\Lambda(J)$ is defined as $\{I \in \Lambda_{\mathrm{diamG_J}} :\ K_I \cap G_J \neq \emptyset\}$; the two definitions do not contain each other.) It is easy to see from the construction that for $I \in \Lambda(J)$ of either type $\mathscr{A}$ or $\mathscr{B}$, $K_I \cap G_J \neq \emptyset$; also K_I and K_J are comparable in size (Lemma 3.1).

For fixed $J_0 \in \mathscr{J}$, the construction of the set $\mathscr{A}$ implies trivially that

$$\Lambda(jJ_0) \supseteq \{jI : I \in \Lambda(J_0)\}, \qquad j = 1, \cdots, N.$$

Our aim is to find J_0 such that the equality holds (Lemma 3.2). In this case the set $\mathscr{B}$ is empty.

3. THE PROOFS

LEMMA 3.1. *There exists $c > 0$ such that $c^{-1} \leq \frac{r_J}{r_I} \leq c$ for all $I \in \Lambda(J),\ J \in \mathscr{J}$.*

Proof. For $I \in \Lambda(J),\ J \in \mathscr{J}$, we consider the two cases:

(i) If $i_1 \neq j_1$, then by the construction in $\mathscr{B}$, we see that $I \in \Lambda_{\mathrm{diamG_J}}$ and by Lemma 2.1(i),

$$r_J \leq \text{diam } G_J \leq r_{i_1 \cdots i_{n-1}} \leq \frac{c_1}{r_{\min}} r_I,$$

where $r_{\min} = \inf_j\{\mathrm{diam} K_j\}$. Also by (2.1) and (2.4) we have

$$r_J \geq (1 + 2c_1\varepsilon)^{-1} \text{diam } G_J \geq (1 + 2c_1\varepsilon)^{-1} r_I.$$

Hence there exists $a > 0$ such that

$$a^{-1} \leq \frac{r_J}{r_I} \leq a. \tag{3.1}$$

(ii) If $i_1 = j_1$, we write

$$J = j_1 \cdots j_l \, j_{l+1} \cdots j_n := j_1 \cdots j_l J', \quad I = j_1 \cdots j_l \, i_{l+1} \cdots i_m := j_1 \cdots j_l I',$$

where $j_{l+1} \neq i_{l+1}$. Then by the construction of $\mathscr{A}$, we see inductively that $I' \in \Lambda(J')$ and by (3.1), $a^{-1} \le r_{J'}/r_{I'} \le a$. This and (2.2) imply that

$$(ac_2^2)^{-1} \le \frac{r_J}{r_I} \le ac_2^2.$$

If we let $c = ac_2^2$, then the lemma follows from the conclusion of the two cases. ∎

LEMMA 3.2. *If in addition* $\{w_j\}_{j=1}^N$ *satisfies the OSC, then* $\gamma = \sup_{J \in \mathscr{J}} \#\Lambda(J) < \infty$. *If we let* $J_0 \in \mathscr{J}$ *such that* $\#\Lambda(J_0) = \gamma$, *then*

$$\Lambda(IJ_0) = \{IJ : J \in \Lambda(J_0)\} \qquad \text{for all } I \in \mathscr{J}. \tag{3.2}$$

Proof. Let U be a bounded open set in the definition of the OSC; then $K \subset \overline{U}$. We claim that there exists $\alpha > 0$ such that for any $x \in K_J$,

$$w_I(U) \subseteq B(x, \alpha r_J) \qquad \text{for all } I \in \Lambda(J). \tag{3.3}$$

Indeed from the construction of $I \in \Lambda(J)$ in $\mathscr{A}$ and $\mathscr{B}$, we have $w_I(K) \cap G_J \neq \emptyset$. Since $w_I(\overline{U}) \supseteq w_I(K)$, we see that $w_I(\overline{U}) \cap G_J \neq \emptyset$. Also by (2.1), there exists $c_3 > 0$ such that

$$r_I \le \operatorname{diam} w_I(\bar{U}) \le c_3 r_I.$$

By (2.4) we have

$$G_J \subseteq B(x, (1 + c_1\varepsilon) r_J).$$

From these we have $w_I(\overline{U}) \subseteq B(x, \alpha r_J)$ for $\alpha = 1 + c_1\varepsilon + c_3$.

Now we observe that $w_I(U)$, $I \in \Lambda(J)$ are disjoint and each contains a ball of radius larger than ar_J for some constant $a > 0$ (by (2.1)). Thus by using (3.3) and a simple volume argument, we conclude that the number of $I \in \Lambda(J)$ is bounded; i.e., $\gamma = \sup_{J \in \mathscr{J}} \Lambda(J) < \infty$.

For (3.2), we have remarked after the definition of $\Lambda(J)$ that $\supseteq$ is trivial. On the other hand, the choice of J_0 implies that $\#\{IJ : J \in \Lambda(J_0)\} = \gamma$. Thus the maximality of γ implies that $\Lambda(IJ_0) = \gamma$ also and (3.2) follows. ∎

THEOREM 3.3. *Suppose* $\{w_j\}_{j=1}^N$ *is a family of contractive, one-to-one self-conformal maps with* $\{|w_j'|\}_{j=1}^N$ *satisfying* (1.1) *and the Dini condition. Then the OSC implies the SOSC.*

Proof. The proof needs only a small modification of [S]; we put it down for completeness. Let $J_0 \in \mathcal{J}$ be chosen as in Lemma 3.2. For any fixed $1 \le l \le N$ and $J \in \mathcal{J}$ with $j_1 \ne l$, we consider the family

$$\mathcal{L} = \{K_L : L \in \Lambda_{\mathrm{diam G_{JJ_0}}} \text{ with } l_1 = l\},$$

where l_1 is the first index of the multiple indices of L. Then $\mathcal{L}$ is a cover of K_l. Since $j_1 \ne l$, (3.2) implies that $L \notin \Lambda(JJ_0)$; hence $K_L \cap G_{JJ_0} = \emptyset$. If we let $D(A, B) = \inf\{|x - y| : \ x \in A, y \in B\}$, then by (2.4), $D(K_L, K_{JJ_0}) \ge c_1^{-1} \varepsilon r_{JJ_0}$, which implies

$$D(K_l, K_{JJ_0}) \ge c_1^{-1} \varepsilon r_{JJ_0} \quad \text{for } l \ne j_1. \tag{3.4}$$

Now let $G_J^* = w_J\big(B(K, \varepsilon/2c_1^2)\big)$ and let

$$U^* = \bigcup_{J \in \mathcal{J}} G_{JJ_0}^*.$$

Then U^* is a bounded open set, $U^* \cap K \ne \emptyset$, and

$$w_j(U^*) = \bigcup_{J \in \mathcal{J}} w_j(G_{JJ_0}^*) = \bigcup_{J \in \mathcal{J}} G_{jJJ_0}^* \subseteq U^*.$$

For $i \ne j$, we claim that $w_i(U^*) \cap w_j(U^*) = \emptyset$. Otherwise, there are I, J such that $G_{iIJ_0}^* \cap G_{jJJ_0}^* \ne \emptyset$. We assume $r_{iIJ_0} \ge r_{jJJ_0}$. Let y be in the intersection; then there exist $y_1 \in K_{iIJ_0}$ and $y_2 \in K_{jJJ_0}$ such that

$$d(y, y_1) \le c_1 \cdot \frac{1}{2c_1^2} \varepsilon \cdot r_{iIJ_0} \le \frac{\varepsilon}{2c_1} r_{iIJ_0}$$

and

$$d(y, y_2) \le c_1 \cdot \frac{1}{2c_1^2} \varepsilon \cdot r_{jJJ_0} \le \frac{\varepsilon}{2c_1} r_{iJJ_0}.$$

Hence

$$D(K_{iIJ_0}, K_j) < c_1^{-1} \varepsilon r_{iIJ_0},$$

which contradicts (3.4) and the proof is complete. ∎

We remark that we can actually prove as in [S] that the OSC is equivalent to $0 < \mathcal{H}^\alpha(K) < \infty$ for a Hausdorff measure $\mathcal{H}^\alpha$. The approach is the same as in [S], modified with this new definition of $\Lambda(J)$ and using the Ruelle operator for the appropriate α [FL].

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