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Multifractal Structure of Convolution of the Cantor Measure

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The multifractal structure of measures generated by iterated function systems (IFS) with overlaps is, to a large extend, unknown. In this paper we study the local dimension of the m-time convolution of the standard Cantor measure μ. By using some combinatoric techniques, we show that the set E of attainable local dimensions of μ contains an isolated point. This is rather surprising because when the IFS satisfies the open set condition, the set E is an interval. The result implies that the multifractal formalism fails without the open set condition. © 2001 Academic Press

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1. INTRODUCTION

Let μ be a probability measure on $\mathbb{R}$. For $s \in \operatorname{supp}\mu$, we define the local dimension $\alpha(s)$ of μ at s by

$$\alpha(s) = \lim_{h\to 0^+} \frac{\log \mu(B_h(s))}{\log h}, \tag{1.1}$$

and let $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$ denote the upper and lower dimension by taking the upper and lower limits. An important consideration in fractal geometry is the multifractal structure of a measure μ generated by an iterated function system (IFS), such as the local dimension spectrum $f(\alpha) = \dim_H K_\alpha$ where $K_\alpha = \{s \in \operatorname{supp}\mu\colon \alpha(s) = \alpha\}$ and the global L^q-scaling spectrum $\tau(q)$. These two classes of spectra are formally governed by the "multifractal formalism" and there is a large amount of literature intended to justify this relationship rigorously (see, for example, [2, 3, 6] and the references therein). The situation is well understood when the IFS satisfies the open set condition, but without that condition very little is known.

Let ν be the standard Cantor measure; then ν can be considered to be generated by the two maps $S_i(x) = \frac{1}{3}x + \frac{2}{3}i$, $i = 0, 1$ with weight $\frac{1}{2}$ on each S_i. Its mth convolution $\mu = \nu * \cdots * \nu$ is generated by

$$S_i(x) = \frac{1}{3}x + \frac{2}{3}i \qquad \text{with weights } 2^{-m}\binom{m}{i}, \qquad i = 0, 1, \ldots, m.$$

It is well known that ν has only one local dimension, namely, $\log 2/\log 3$. For $\mu = \nu * \nu$, the IFS $\{S_i\}_{i=0}^2$ satisfies the open set condition; there is an explicit formula for the L^q-scaling spectrum $\tau(q)$ and the local dimension spectrum $f(\alpha)$ can be obtained by the multifractal formalism ($f(\alpha)$ equals the Legendre transformation (concave conjugate) of $\tau(q)$). For the m-time convolution the IFS $\{S_i\}_{i=0}^m$ does not satisfy the open set condition. In [4], Fan, Lau, and Ngai had made an initial investigation on the multifractal structure of such measure. They provided an algorithm to calculate the L^q-scaling spectrum $\tau(q)$ for q positive integers. By using the multifractal formalism, they obtained some approximation of $f(\alpha)$ for the α corresponding to $\tau'(q)$, $q > 0$. However, nothing is known for the rest of the $f(\alpha)$.

Let $E = \{\alpha\colon \alpha(s) = \alpha \text{ for some } s \in \operatorname{supp}\mu\}$ be the set of attainable local dimensions. In this paper we show that

THEOREM 1.1. *Let μ be the mth convolution of the Cantor measure ($m \geq 3$). Then $\overline{\alpha} = \sup\{\overline{\alpha}(s)\colon s \in \operatorname{supp}\mu\} = \frac{m\log 2}{\log 3}$ is an isolated point of E.*

For the case $m = 3$ we have a more precise result.

THEOREM 1.2. *Let μ be the three-time convolution of the Cantor measure. Then*

(i) $\underline{\alpha} = \inf\{\underline{\alpha}(s)\colon s \in \operatorname{supp}\mu\} = \frac{3\log 2}{\log 3} - 1 \approx 0.89278$; $\overline{\alpha} = \sup\{\overline{\alpha}(s)\colon s \in \operatorname{supp}\mu\} = \frac{3\log 2}{\log 3} \approx 1.89278$.

(ii) $E = [\underline{\alpha}, \tilde{\alpha}] \cup \{\overline{\alpha}\}$ *with* $\tilde{\alpha} = \frac{3\log 2}{\log 3} - \frac{\log b}{2\log 3} \approx 1.1335$ *where* $b = \frac{7+\sqrt{13}}{2}$.

In order for the multifractal formalism to hold, $f(\alpha)$ must be a concave function and the domain is an interval; i.e., the set of local dimensions of α forms an interval. This is true for all self-similar measures (actually more general) generated by IFS satisfying the open set condition [2, 6]. The above conclusion (ii) implies that the multifractal formalism fails for the convolution of the m-time convolution ($m \geq 3$) of the Cantor measure μ at least at $\overline{\alpha}$. Nevertheless, the formalism may still hold excluding $\overline{\alpha}$.

The proof of the theorems is combinatoric; it depends on some careful counting of the multiple representations of $s = \sum_{j=1}^{\infty} 3^{-j}x_j$, $x_j = 0, \ldots, m$, and the associated probability. We remark that there are recent investigations of the tenary expansions and other λ-expansions in connection with the fractal structure of the underlying sets [8–11].

In Section 2 we will give some preliminaries and prove some basic lemmas for counting. In Section 3 we prove Theorem 1.1 among the other results (Theorem 3.2, Theorem 3.6). In Section 4 we calculate the precise local dimensions and $\tilde{\alpha}$ for $m = 3$ as stated in Theorem 1.2.

2. THE BASIC LEMMAS

Let ν be the standard Cantor measure and let $\mu = \nu * \cdots * \nu$ (m-times). Note that μ can be obtained in the following way: Let X be a random variable taking values $\{0, 1, \ldots, m\}$ with probability

$$p_i = P(X = i) = \frac{1}{2^m}\binom{m}{i}$$

and let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the same distribution as X. Let $S = \sum_{j=1}^{\infty} 3^{-j}X_j$. Then the range of S is $[0, \frac{m}{2}]$. Let $S_n = \sum_{j=1}^{n} 3^{-j}X_j$; then S_n takes values $s_n \in \{3^{-n}i\colon i = 0, 1, \ldots, m(3^n - 1)/2\}$. Let μ_n and μ be the distribution measure of S_n and S, respectively.

LEMMA 2.1. *Suppose $m \geq 2$; then for any two consecutive $s_n = 3^{-n}i$, $s_n' = 3^{-n}(i-1)$, we have*

$$\frac{1}{(n+1)\theta} \leq \frac{\mu_n(s_n')}{\mu_n(s_n)} \leq (n+1)\theta,$$

where

$$\theta = \binom{m}{[\frac{m+1}{2}]} = \max_{1\le i\le m} p_i / \min_{1\le i\le m} p_i.$$

Proof. It is clear that the lemma is true for $n = 1$. Suppose it is true for $n = k$. Consider $n = k+1$; then there is an integer r such that $s_{k+1} = \sum_{j=1}^{k+1} 3^{-j} x_j = 3^{-k} r + 3^{-(k+1)} x_{k+1}$. We can write

$$s_{k+1} = 3^{-k}(r-j) + 3^{-(k+1)}(x_{k+1} + 3j),$$

where $r - j \ge 0$ and $0 \le x_{k+1} + 3j \le m$. Denote this set of j by J_1. It follows that

$$\mu_{k+1}(s_{k+1}) = \sum_{j\in J_1} \mu_k\big(3^{-k}(r-j)\big) P(X = x_{k+1} + 3j).$$

Similarly the preceding value $s'_{k+1} = s_{k+1} - 3^{-(k+1)}$ satisfies

$$\mu_{k+1}(s'_{k+1}) = \sum_{j\in J_2} \mu_k\big(3^{-k}(r-j)\big) P(X = (x_{k+1} - 1) + 3j),$$

where J_2 is the set of j such that $0 \le x_{k+1} - 1 + 3j \le m$. Note that

$$j \in J_l \qquad \text{if and only if } \frac{-x_{k+1} + \epsilon_l}{3} \le j \le \frac{m - x_{k+1} + \epsilon_l}{3},$$

where $\epsilon_1 = 0$ and $\epsilon_2 = 1$, $l = 1, 2$. There are three possibilities: (a) $J_1 \subset J_2$, (b) $J_2 \subset J_1$, and (c) $J_1 = J_2$. In case (a), $j' = (m - x_{k+1} + 1)/3$ is the only integer contained in $J_2 \backslash J_1$ and $P(X = (x_{k+1} - 1) + 3j') = P(X = m) = \min_{0\le i\le m} p_i$. Then

$$\begin{aligned}
\frac{\mu_{k+1}(s'_{k+1})}{\mu_{k+1}(s_{k+1})} &\le \frac{\big(\max\limits_{0\le i\le m} p_i\big) \sum_{j\in J_1} \mu_k(3^{-k}(r-j)) + \mu_k(3^{-k}(r-j'))P(X=m)}{\big(\min\limits_{0\le i\le m} p_i\big) \sum_{j\in J_1} \mu_k\big(3^{-k}(r-j)\big)} \\
&= \theta + \frac{\mu_k\big(3^{-k}(r-j')\big)}{\sum_{j\in J_1} \mu_k\big(3^{-k}(r-j)\big)} \\
&\le \theta + \frac{\mu_k\big(3^{-k}(r-j')\big)}{\mu_k\big(3^{-k}(r-(j'-1))\big)} \\
&\le \theta + k\theta \qquad \text{(by induction)} \\
&= (k+1)\theta.
\end{aligned}$$

A similar proof implies that the lower bound of the quotient is $\frac{1}{(k+1)\theta}$. Case (b) follows by the same argument and case (c) is trivial. ∎

PROPOSITION 2.2. *Let $m \geq 2$; then*

$$\alpha(s) = \lim_{n\to\infty} \left|\frac{\log \mu_n(s_n)}{n \log 3}\right|,$$

provided that the limit exists. Otherwise we can replace $\alpha(s)$ by $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$ and consider the upper and the lower limits.

Proof. By definintion it is clear that

$$\alpha(s) = \lim_{n\to\infty} \left|\frac{\log \mu(B_{3^{-n}}(s))}{n \log 3}\right|.$$

Also it is easy to prove that

$$\mu\big(B_{3^{-n}}(s)\big) \leq \mu_n\big(B_{r3^{-n}}(s)\big) \leq \mu\big(B_{2r3^{-n}}(s)\big)$$

for $r = 1 + \frac{m}{2}$ and $B_{r3^{-n}}(s)$ contains at most $[2r]$ consecutive s_n ($[x]$ is the greatest integer $\leq x$). The proposition hence follows from the lemma. ∎

It is clear that if $m \geq 3$, then the series representation $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j$ is not unique. In the following we will prove a key lemma concerning the multiple representations of s. It will be used throughout the paper.

PROPOSITION 2.3. *Let $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j$, $s' = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$, and $s - s' = \Sigma_{j=1}^{\infty} 3^{-j} y_i$.*

(i) *If $s_n = s'_n$, then $x_n \equiv x'_n$ (mod 3). If, further, we assume that $|y_j| \leq 3$ for all j, then $(y_1, \ldots, y_n)$ can be decomposed as segments of the forms*

$$(0, 0, \ldots, 0), \qquad \pm(-1, 3) \qquad \text{and} \qquad \pm(-1, 2, \ldots, 2, 3). \tag{2.1}$$

(ii) *Conversely if $(y_1, \ldots, y_n, \ldots)$ can be decomposed as segments as in (2.1) or $\pm(-1, 2, 2, \ldots)$, then $s = s'$.*

Proof. To prove the first statement in (i), we multiply 3^n to $s_n - s'_n = 0$. It follows that

$$3^n(x_1 - x'_1) + \cdots + 3(x_{n-1} - x'_{n-1}) + (x_n - x'_n) = 0$$

and hence $x_n \equiv x'_n$ (mod 3). For the second statement in (i), we note that the last non-zero term of $y_1, \ldots, y_n$ must be congruent to 0 module 3. Since $|y_j| \leq 3$, we can assume without loss of generality that $y_n = 3$. Hence by rewriting $s_n = s'_n$ as

$$\sum_{j=1}^{n-2} 3^{-j} y_j + 3^{-(n-1)}(y_{n-1} + 1) = 0, \tag{2.2}$$

we see that $y_{n-1} + 1 \equiv 0$ (mod 3). Since $|y_j| \leq 3$, either $y_{n-1} = -1$ or 2. If $y_{n-1} = -1$, then $(y_{n-1}, y_n) = (-1, 3)$ as asserted. We repeat the same argument to $\Sigma_{j=1}^{n-2} 3^{-j} y_j = 0$. If $y_{n-1} = 2$, then we can write (2.2) as $\Sigma_{j=1}^{n-2} 3^{-j} y_j + 3^{-(n-2)} = 0$ so that

$$\sum_{j=1}^{n-3} 3^{-j} y_j + 3^{-(n-2)}(y_{n-2} + 1) = 0.$$

This is the same form as (2.2) and the process can be repeated. The proof for (ii) is trivial. ∎

We conclude this section by introducing some notations. For $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j$, we write the digits by the vector $\mathbf{x} = (x_1, x_2, \ldots)$ and let $\langle s \rangle$ be the equivalent class of the $\mathbf{x}' = (x_1', x_2', \ldots)$ such that $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j'$. Also we let

$$\langle s_n \rangle = \left\{ (x_1', \ldots, x_n'):\ s_n = \sum_{j=1}^{n} 3^{-j} x_j' \right\}.$$

3. THE EXTREME LOCAL DIMENSIONS

In this section, we assume that μ is the mth convolution of the Cantor measure, $m \geq 2$. Then μ is supported by $[0, \frac{m}{2}]$. Let

$$\overline{\alpha} = \sup\{\overline{\alpha}(s):\ s \in \operatorname{supp} \mu\} \qquad \text{and} \qquad \underline{\alpha} = \inf\{\underline{\alpha}(s):\ s \in \operatorname{supp} \mu\}.$$

PROPOSITION 3.1. *For $m \geq 2$, $\overline{\alpha} = \frac{m \log 2}{\log 3}$ and the value is attained at $s = 0$ or $\frac{m}{2}$.*

Proof. Let $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j \in [0, \frac{m}{2}]$. Then

$$\mu_n(s_n) \geq \prod_{j=1}^{n} P(X = x_j) \geq 2^{-mn}.$$

It follows from Proposition 2.2 that

$$\overline{\alpha}(s) = \overline{\lim_{n\to\infty}} \left| \frac{\log \mu_n(s_n)}{n \log 3} \right| \leq \overline{\lim_{n\to\infty}} \left| \frac{\log 2^{-mn}}{n \log 3} \right|.$$

On the other hand consider $s = 0$ or $s = \frac{m}{2}$; they have unique digit representation $(0, 0, \ldots)$ or $(m, m, \ldots)$ respectively. By Proposition 2.2 and a direct calculation, it is clear that $\overline{\alpha}(s) = \frac{m \log 2}{\log 3}$. ∎

Let $K(\alpha) = \{s \in \operatorname{supp} \mu:\ \alpha(s) = \alpha\}$, i.e., the set of points s such that the local dimension of μ at s is α.

THEOREM 3.2. *Let $m \geq 3$. Then*

(i) $K(\overline{\alpha}) = \{0, \frac{m}{2}\}$.

(ii) $K(\alpha) = \phi$ *for all* $\alpha^* < \alpha < \overline{\alpha}$, *where*

$$\alpha^* = \begin{cases} \dfrac{m \log 2}{\log 3} - \dfrac{\log \binom{m}{m/2-1}}{\log 3}, & \text{if } m \text{ is even}, \\[2ex] \dfrac{m \log 2}{\log 3} - \dfrac{\log \binom{m}{(m+1)/2} + \log \binom{m}{(m+1)/2-2}}{2 \log 3}, & \text{if } m \text{ is odd}. \end{cases}$$

The unexpected part of the theorem is that there is no point the local dimension of μ is between α^* and $\overline{\alpha}$. Note that (ii) of the theorem is not true for $m = 2$ (see [4]). We need a few technical lemmas to prove the theorem. The main idea is that for any s other than 0 and $\frac{m}{2}$, we can find another representation with digits around the middle of $0, 1, \ldots, m$ so that s will associate with a heavier weight. The local dimension can be computed to be much smaller than $\overline{\alpha}$ and hence produces a gap there. The first one is a pre-lemma of Lemma 3.4.

LEMMA 3.3. *Let $m \geq 3$ and let $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j \in (0, \frac{m}{2})$. Then for any fixed $3 \leq r \leq m$, there exists k and another representation $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ such that $0 \leq x'_j \leq r - 1$ for all $j \geq k$.*

Proof. We assume the contrary; let $q = \max\{x_j\colon j > 1\} \geq r$, $x_1 \neq q$ and there are infinitely many $x_j = q$. We can use the following procedure repeatedly to reduce the size of q until $q \leq r - 1$.

(i) There exists i_0 such that $x_j = q$ for all $j > i_0$ and $x_{i_0} < q$. Let

$$x'_{i_0} = x_{i_0} + 1, \qquad x'_j = x_j - 2 \qquad \forall j > i_0, \qquad \text{and} \qquad x'_j = x_j \qquad \forall j < i_0.$$

Then by Proposition 2.3(ii), $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ and $\max_{j > i_0}\{x'_j\} = q - 2$.

(ii) If $x_j < q$ for infinitely many j, we can assume without loss of generality that $x_1 < q - 1$ (this condition will appear in the following x'_n in the second iteration) and let n be the smallest integer such that $x_n = q$. Let i_0 be the largest integer less than n such that $x_{i_0} < q - 1$. Let

$$x'_{i_0} = x_{i_0} + 1, \qquad x'_n = x_n - 3, \qquad x'_j = x_j - 2 \qquad \text{for} \qquad i_0 < j \leq n - 1,$$

and $x'_j = x_j$ otherwise. Then $s = \Sigma_{n=1}^{\infty} 3^{-j} x'_j$ by Proposition 2.3(ii) and $0 \leq x'_j \leq q - 1$, for $1 \leq j \leq n$. We will repeat this procedure to have all $x_j \leq q - 1$. ∎

LEMMA 3.4. *Suppose that $m \geq 3$ and $0 \leq r \leq m - 2$. Let $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j \in (0, \frac{m}{2})$; then there exists k and another representation $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ such that $x'_j \in \{r, r+1, r+2\}$ for all $j \geq k$.*

Proof. By Lemma 3.3 (apply to $r + 3$) we can assume without loss of generality that $0 \leq x_j \leq r + 2$ for all j. If $r = 0$, the lemma is automatic. Hence we assume that $r > 0$; we show that we can replace $x_j = 0$ by $1 \leq x'_j \leq r + 2$, where $r \geq 1$. Assume that there exist some $x_n = 0$. We need to deal with two cases.

(i) If $x_j = 0$ or 1 for all j, let $j_0 = \min\{j\colon x_j = 1\}$. Define

$$x'_j = 0 \quad \text{for } j \leq j_0, \qquad x'_j = x_j + 2 \quad \text{for} \quad j > j_0.$$

Then $2 \leq x'_j \leq 3 \leq r + 2$ for $j > j_0$ and $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ by Proposition 2.3.

(ii) Otherwise consider a segment of the form $(x_i, x_{i+1}, \ldots, x_n)$ with $x_i > 1, x_{i+1}, \ldots, x_{n-1} = 0$ or 1 and $x_n = 0$. We define

$$x'_i = x_i - 1, \qquad x'_n = x_n + 3, \qquad \text{and} \qquad x'_j = x_j + 2, \qquad i < j \leq n - 1.$$

Then $s = \Sigma_{j=1}^{n} 3^{-j} x'_j$ and $x'_j > 0$ for $i \leq j \leq n$. We repeat this process until all the 0 after x_n are replaced.

After we have $1 \leq x'_j \leq r + 2$ we can repeat the same process until we obtain a representation $s = \Sigma_{j=1}^{\infty} 3^{-j} x''_j$ with $r \leq x''_j \leq r + 2$. ∎

LEMMA 3.5. *Suppose $m \geq 3$ and $0 \leq r \leq m - 2$. Let $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j \in (0, \frac{m}{2})$; then there exists k and another representation $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ satisfying*

(i) $x'_j \in \{r, r+1, r+2, r+3\}$ *for all* $j \geq k$.

(ii) *For any* $j \geq k$, $(x'_j, x'_{j+1}) \neq (r, r), (r, r+3), (r+3, r), (r+3, r+3)$.

Consequently if n' is the total number of digits x'_j such that $x'_j = r + 1$ or $r + 2$, then $n' \geq (n - k)/2$.

Proof. By Lemma 3.4, we can assume that $x_j \in \{r, r+1, r+2\}$. For convenience we also let $r = 0$ so that $x_j = 0, 1$ or 2. We need to replace the segments $(x_j, \ldots, x_{j+k}) = (0, \ldots, 0), k \geq 1$, to satisfy conditions (i) and (ii).

Without loss of generality we assume that $x_1 \neq 0$. Let $(x_{i_0}, \ldots, x_{i_0+k})$ be the first segement of 0 with x_{i_0-1} and $x_{i_0+k+1} \geq 1$. If $(x_1, x_2, \ldots, x_{i_0-1}) = (1, 0, 1, 0, \ldots, 1, 0, 1)$, then let $j_0 = 1$. Otherwise $(x_1, x_2, \ldots, x_{i_0-1})$ contains a 2 or $(1, 1)$; we let

$$j_0 = \max\{j < i_0 \colon x_j = 2 \text{ or } (x_{j-1}, x_j) = (1, 1)\}.$$

(Note that for $j_0 < j < i_0$, the digits x_j are alternative 0 and 1.) We define

$$x'_{j_0} = x_{j_0} - 1, \quad x'_j = x_j + 2 \qquad \text{for } j_0 < j \leq i_0 + k - 1, \quad x'_{i_0+k} = 3$$

and $x'_j = x_j$ otherwise. Then $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ and for $1 \leq j \leq i_0 + k$, x'_j satisfy conditions (i) and (ii) of the lemma. We repeat this argument for $j > i_0 + k$ and (i) and (ii) of the lemma will follow. The second part is clear. ∎

Proof of Theorem 3.2. If m is even, then by Lemma 3.4, there exists k and another series representation $s = \Sigma_{j=1}^{\infty} 3^{-j} x'_j$ such that $x'_j \in \{\frac{m}{2} - 1, \frac{m}{2}, \frac{m}{2} + 1\}$ for all $j \geq k$. It follows that

$$\mu_n(s_n) \geq \prod_{j=1}^{n} P(X = x'_j) \geq C\left(2^{-m}\binom{m}{\frac{m}{2} - 1}\right)^n$$

(C depends on k) so that

$$\overline{\alpha}(s) = \overline{\lim_{n\to\infty}} \left| \frac{\log \mu_n(s_n)}{n \log 3} \right| \le \alpha^*.$$

If m is odd, we take $x'_j \in \{r, r+1, r+2, r+3\}$ where $r = \frac{m+1}{2} - 2$; then by Lemma 3.5,

$$\mu_n(s_n) \ge \prod_{j=1}^{n} P(X = x'_j) \ge C\left(2^{-m}\binom{m}{\frac{m+1}{2}}\right)^{n/2}\left(2^{-m}\binom{m}{\frac{m+1}{2}-2}\right)^{n/2}$$

and $\overline{\alpha}(s) \le \alpha^*$. Now (i) and (ii) follow from this and Proposition 3.1. ∎

Our second theorem is concerned with the smallest local dimension $\underline{\alpha}$. We can prove it only for the case $m \le 4$.

THEOREM 3.6. *Let $2 \le m \le 4$. Then*

$$\underline{\alpha} = \begin{cases} \dfrac{3\log 2}{\log 3} - 1 \approx 0.89278 & \text{if } m = 3 \text{ or } 4, \\ \dfrac{\log 2}{\log 3} \approx 0.63093 & \text{if } m = 2. \end{cases}$$

Moreover the infimum is attained at $s = \sum_{j=1}^{\infty} 3^{-j} = \frac{1}{2}$ if $m = 2$; $s = \sum_{j=1}^{\infty} 3^{-j}2 = 1$ if $m = 4$; and $s = \sum_{j=1}^{\infty} 3^{-j}x_j,\ \ x_j = 1$ or 2 if $m = 3$.

Proof. We will prove the theorem for $m = 4$. The case for $m = 3$ and $m = 2$ can be handled in the same way. Let $t = \sum_{j=1}^{\infty} 3^{-j}2$. We claim that $\langle t_n \rangle = \{(2, \ldots, 2)\}$. Indeed for $(x_1, \ldots, x_n) \in \langle t_n \rangle$, by Proposition 2.3(i), $x_n - t_n \equiv 0 \pmod 3$; hence $x_n = 2$ also. Thus $(x_1, \ldots, x_{n-1}) \in \langle t_{n-1} \rangle$ and a simple induction implies that $x_i = 2,\ 1 \le i \le n$. Hence

$$\mu_n(t_n) = \left(2^{-4}\binom{4}{2}\right)^n = \left(\frac{6}{2^4}\right)^n$$

and

$$\underline{\alpha}(t) = \lim_{n\to\infty} \left| \frac{\log \mu_n(t_n)}{n \log 3} \right| = \frac{3\log 2}{\log 3} - 1 = \underline{\alpha}.$$

It remains to show that for any $s = \sum_{j=1}^{\infty} 3^{-j}x_j$, $\mu_n(s_n) \le \mu_n(t_n)$ so that $\underline{\alpha}(s) \ge \underline{\alpha}$. We will prove this by induction. For the case $n+1$,we divide it into three cases:

(i) If $x_{n+1} = 2$, then

$$\mu_{n+1}(s_{n+1}) = \mu_n(s_n)P(X = 2) \leq \left(\frac{6}{2^4}\right)^n \left(\frac{6}{2^4}\right) = \mu_{n+1}(t_{n+1}).$$

(ii) If $x_{n+1} = 0$ (or 3), then by Proposition 2.3(i), for any other representation $s_{n+1} = \Sigma_{j=1}^{n+1} 3^{-j} x'_j$, x'_{n+1} has two choices: 0 or 3. Let $s_n^{(1)}, s_n^{(2)}$ be the corresponding n-sum of the two choices; then

$$s_{n+1} = s_n^{(1)} + 3^{-(n+1)}0 = s_n^{(2)} + 3^{-(n+1)}3.$$

By the induction hypothesis we have

$$\begin{aligned} \mu_{n+1}(s_{n+1}) &= \mu_n(s_n^{(1)})P(X = 0) + \mu_n(s_n^{(2)})P(X = 3) \\ &= \mu_n(s_n^{(1)})\left(\frac{1}{2^4}\right) + \mu_n(s_n^{(2)})\left(\frac{4}{2^4}\right) \\ &\leq \mu_n(t_n)\left(\frac{5}{2^4}\right) = \left(\frac{6}{2^4}\right)^n \left(\frac{5}{2^4}\right) \\ &< \left(\frac{6}{2^4}\right)^{n+1} = \mu_{n+1}(t_{n+1}). \end{aligned}$$

(iii) If $x_{n+1} = 1$ (or 4), the proof is the same as (ii) since $P(X = 0) = P(X = 4)$ and $P(X = 1) = P(X = 3)$. ∎

4. THE EXACT RANGE OF LOCAL DIMENSION: $m = 3$

In this section we only consider that μ is the three time convolution of the standard Cantor measure. In last section we showed that there is no $s \in \operatorname{supp} \mu$ with local dimension $\alpha \in (\alpha^*, \overline{\alpha})$ where $\alpha^* = \frac{3 \log 2}{\log 3} - \frac{1}{2}$ and $\overline{\alpha} = \frac{3 \log 2}{\log 3}$. However, such α^* is not the best possible value. We will sharpen this result in the sequel.

LEMMA 4.1. *Let* $0 < \beta_1, \beta_2$ *be fixed and let* $\beta_{n+1} = 4\beta_n + \Sigma_{j=1}^{n-1} 3^{n-j} \beta_j$. *Then*

(i) $\beta_{n+1} = 7\beta_n - 9\beta_{n-1}$.

(ii) $\beta_{n+1} = cb^n + c'b'^n$ *where* $b, b' = \frac{7 \pm \sqrt{13}}{2}$ (≈ 5.3, 1.7, *respectively*) *are roots of* $x^2 - 7x + 9 = 0$, *and* c *and* c' *depend on* β_1 *and* β_2.

Proof. By definition,

$$\beta_{n+1} = 4\beta_n + 3\left(\beta_{n-1} + \sum_{j=1}^{n-2} 3^{n-1-j}\beta_j\right) = 4\beta_n + 3(\beta_n - 3\beta_{n-1}).$$

So (i) follows. It is easy to show that $\{\beta_n\}$ is an increasing sequence and $b = \lim_{n\to\infty} \beta_{n+1}/\beta_n$ satisfies $b^2 - 7b + 9 = 0$. The expression of β_n in (ii) follows from [1, Chap. 6]. ∎

We remark that c, c' are uniquely determinined by the two equations $c + c' = \beta_1, cb + c'b' = \beta_2$; i.e.,

$$c = \frac{\beta_2 - \beta_1 b'}{b - b'} \qquad \text{and} \qquad c' = \frac{\beta_1 b - \beta_2}{b - b'}.$$

In the later part we will have $\beta_2 \geq 4\beta_1$ so that c is always positive, but c' can be positive or negative. If $c' > 0$, then $\beta_{n+1} > cb^n$. If $c' < 0$, then $\beta_{n+1} \geq (c + c')b^n = \beta_1 b^n$.

The calculation in this section depends very much on the following special t and its variations.

LEMMA 4.2. *Let* $t = \Sigma_{j=1}^{\infty} 3^{-j} x_j = \frac{1}{8}$ *with* $\mathbf{x} = (0, 1, 0, 1, \ldots)$. *Then*

$$\mu_{2n-1}(t_{2n-1}) = \frac{\beta_n}{8^{2n-1}}, \qquad \mu_{2n}(t_{2n}) = \frac{3\beta_n}{8^{2n}},$$

where β_n *is defined as in Lemma* 4.1 *with* $\beta_1 = 1, \beta_2 = 4$. *In this case* $\beta_n \geq 0.63b^{n-1}$.

Proof. For $n = 1$, $\mu_1(t_1) = \frac{1}{8} = \beta_1/8$. For $n = 2$, by Proposition 2.3(i) we see that t_3 has two representations: $\langle t_3 \rangle = \{(0, 1, 0), (0, 0, 3)\}$. Therefore

$$\mu_3(t_3) = p_0 p_1 p_0 + p_0 p_0 p_3 = \frac{4}{8^3} = \frac{\beta_2}{8^3}.$$

For n we observe that for $(x'_1, \ldots, x'_{2n-1}) \in \langle t_{2n-1} \rangle$, Proposition 2.3(i) implies that $0 - x'_{2n-1} \equiv 0 \pmod 3$. We have the following two cases for x'_{2n-1}.

(a) $x'_{2n-1} = 3$: By Proposition 2.3(ii) we can find an i such that

$$(x_i - x'_i, \ldots, x_{2n-1} - x'_{2n-1}) = (1, -2, \ldots, -2, -3).$$

Note that $x_i - x'_i = 1 - x'_i = 1$ implies that $x_i = 1, x'_i = 0$; i.e., i is an even integer and hence

$$(x'_i, \ldots, x'_{2n-1}) = (0, 2, 3, 2, 3, \ldots, 2, 3, 3).$$

Let $i = 2k$; then the digit 2 occurs $(n - k - 1)$ times, 3 occurs $(n - k)$ times, and 0 occurs once.

(b) $x'_{2n-1} = 0$: By Proposition 2.3(ii), the preceding term x'_{2n-2} must be 1 so that $t_{2n-3} = t'_{2n-3}$.

We see that t_{2n-1} has only two representations as in (a), (b); hence

$$\mu_{2n-1}(t_{2n-1}) = \mu_{2n-3}(t_{2n-3})p_1p_0 + \sum_{k=1}^{n-1} \mu_{2k-1}(t_{2k-1})p_0p_2^{n-k-1}p_3^{n-k}$$

$$= \frac{3\beta_{n-1}}{8^{2n-1}} + \sum_{k=1}^{n-1} 3^{(n-k-1)} \frac{\beta_k}{8^{2n-1}} = \frac{\beta_n}{8^{2n-1}}.$$

This completes the induction for the first equality. For the even case we observe that $t_{2n} = t_{2n-1} + 3^{-2n}$ is the only representation for s_{2n}. This implies that

$$\mu_{2n}(t_{2n}) = \mu_{2n-1}(t_{2n-1})\frac{3}{8} = \frac{3\beta_n}{8^{2n}}.$$

The last inequality follows from the remark after Lemma 4.1 with $c \geq 0.63$. ∎

COROLLARY 4.3. *Let* $\mathbf{x}_t = (0, 1, 0, 1, \ldots)$ *and for* $\mathbf{x} = (x_1, x_2, \ldots)$ *let* $s = \sum_{j=1}^{\infty} 3^{-j}x_j$

(i) *If* $\mathbf{x} = (2, \mathbf{x}_t)$, *then*

$$\frac{6b^{n-1}}{8^{2n}} \leq \mu_{2n}(s_{2n}) \leq \frac{7b^{n-1}}{8^{2n}}.$$

(ii) *If* $\mathbf{x} = (2, 3, 1, \mathbf{x}_t)$, *then*

$$\frac{4b^{n-1}}{8^{2n}} \leq \mu_{2n}(s_{2n}) \leq \frac{7b^{n-1}}{8^{2n}}.$$

(iii) *If* $\mathbf{x} = (1, 1, \mathbf{x}_t), (2, 1, \mathbf{x}_t), (1, 2, \mathbf{x}_t)$ *or* $(2, 2, \mathbf{x}_t)$, *then*

$$\frac{2b^n}{8^{2n+1}} \leq \mu_{2n+1}(s_{2n+1}) \leq \frac{4b^n}{8^{2n+1}}.$$

Proof. (i) Consider $\mathbf{x} = (2, \mathbf{x}_t) = (2, 0, 1, 0, \ldots)$. s_2 has two representations, $(2, 0)$ and $(1, 3)$, so that

$$\mu_2(s_2) = \frac{3+3}{8^2} = \frac{6}{8^2} := \frac{\beta_1}{8^2}.$$

s_4 has five representations, $(2, 0, 1, 0)$, $(1, 3, 1, 0)$, $(2, 0, 0, 3)$, $(1, 3, 0, 3)$, $(1, 2, 3, 3)$; hence

$$\mu_4(s_4) = \frac{9+9+3+3+9}{8^4} = \frac{33}{8^4} := \frac{\beta_2}{8^4}.$$

Now using the same argument as in Lemma 4.2, we have $\mu_{2n}(s_{2n}) = \beta_n/8^{2n}$, where $\beta_1 = 6, \beta_2 = 33$. We can calculate $c' \approx -0.33, c \approx 6.33$ as in Lemma 4.1. By the remark after Lemma 4.1, we have $6b^{n-1} \leq \beta_n \leq 7b^{n-1}$ and (i) follows.

(ii) For $\mathbf{x} = (2, 3, 1, \mathbf{x}_t)$, s_2 has two representations, $(2, 3)$ and $(3, 0)$; s_4 has five representations, $(2, 3, 1, 0)$, $(2, 3, 0, 3)$, $(3, 0, 1, 0)$, $(3, 0, 0, 3)$, and $(2, 2, 3, 3)$. Hence we have $\beta_1 = 4$, $\beta_2 = 25$, and we can show that $c \approx 5.05$, $c' \approx -1.05$. This implies that $4b^{n-1} \le \beta_n \le 7b^{n-1}$.

(iii) The proof is similar. ∎

THEOREM 4.4. *Let* $\tilde{\alpha} = \frac{3\log 2}{\log 3} - \frac{\log b}{2\log 3} \approx 1.1335$. *Then* $K(\alpha) = \phi$ *for all* $\tilde{\alpha} < \alpha < \overline{\alpha}$. *Moreover the value* $\tilde{\alpha}$ *is attained at* $t = \Sigma_{j=1}^{\infty} 3^{-i} x_j$ *(i.e.,* $\alpha(t) = \tilde{\alpha}$*) where* $(x_1, x_2, \ldots) = (0, 1, 0, 1, \ldots)$.

REMARK. *This improves Theorem* 3.2 *for* $m = 3$ *where* $\alpha^* = \frac{3\log 2}{\log 3} - \frac{1}{2} \approx 1.39278$.

Proof. Let $t = \Sigma_{j=1}^{\infty} 3^{-j} x_j$ where $(x_1, x_2, \ldots) = (0, 1, 0, 1, \ldots)$. That $\alpha(t) = \tilde{\alpha}$ is a direct consequence of Lemma 4.2. We claim that for any $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j$ with $x_j = 0, 1, 2, 3$, there is a constant c depending on s such that

$$\mu_{2n}(s_{2n}) \ge \frac{cb^n}{8^{2n}} \qquad \text{and} \qquad \mu_{2n+1}(s_{2n+1}) \ge \frac{cb}{3}\frac{b^n}{8^{2n+1}}. \tag{4.1}$$

This will imply $K(\alpha) = \phi$ for $\tilde{\alpha} < \alpha < \overline{\alpha}$.

To prove the claim, we can assume, by Lemma 3.4, that $x_j = 0, 1$, or 2. For convenience we assume further that $x_1 = 1$ (or 2) and $c = \frac{1}{2}$ (otherwise, we can start from the first non-zero term and adjust the constant c). We will prove the statement by two inductive steps:

Step 1. We show that for $1 \le n \le n_0$,

$$\mu_{2n}(s_{2n}) \ge \frac{cb^n}{8^{2n}} \quad \Rightarrow \quad \mu_{2n+1}(s_{2n+1}) \ge \frac{cb}{3}\frac{b^n}{8^{2n+1}}.$$

It is straightforward to verify this for $n = 1$. Suppose it is true for $n = k - 1$ and consider $n = k$. If the final digit x_{2k+1} is 1 or 2, then

$$\mu_{2k+1}(s_{2k+1}) = \mu_{2k}(s_{2k}) p_1 \ge \frac{cb^k}{8^{2k}}\frac{3}{8} \ge \frac{cb}{3}\frac{b^k}{8^{2k+1}}.$$

If the final digit $x_{2k} = 0$, then we run the digits 0 and 1 alternatively backward and stop at i until one of the following cases occurs:

(i) $(x_1, \ldots, x_i, 2, 0, 1, \ldots, 0, 1, 0)$;

(ii) $(x_1, \ldots, x_i, 1, 0, 1, \ldots, 0, 1, 0)$, where $x_i \ne 0$;

(iii) $(x_1, \ldots, x_i, 0, \ldots, 0, 1, \ldots, 0, 1, 0)$ where $x_i \ne 0$ and there are at least two consecutive zeros starting from the $(i + 1)$th term.

In case (i) we see that i is odd. Write $i = 2j+1$; then by Corollary 4.3(i) and induction,

$$\mu_{2k+1}(s_{2k+1}) \geq \mu_{2j+1}(s_{2j+1})\frac{6b^{k-j-1}}{8^{2(k-j)}} > \frac{cb}{3}\frac{b^j}{8^{2j+1}}\frac{6b^{k-j-1}}{8^{2(k-j)}} > \frac{cb}{3}\frac{b^k}{8^{2k+1}}.$$

In case (ii), $i = 2j+1$ for some j; we divide the digits into two segments $(x_1, \ldots, x_{i-1})$ and $(x_i, 1, 0, 1, \ldots, 1, 0)$. Then

$$\mu_{2k+1}(s_{2k+1}) \geq \mu_{2j}(s_{2j})\frac{2b^{k-j}}{8^{2(k-j)+1}} \geq \frac{cb^j}{8^{2j}}\frac{2b^{k-j}}{8^{2(k-j)+1}} > \frac{cb}{3}\frac{b^k}{8^{2k+1}}.$$

In (iii) we consider the case with $x_i = 1$ and only two consecutive zeros after x_i (for the case of more zeros or $x_i = 2$, the proof is the same). It is clear that i is odd. Write $i = 2j+1$. There are at least two representations as follows. We divide the digits into two subcases (by parentheses) for calculation:

(a) $(x_1, \ldots, x_{i-1}, 1, 0)(0, 1, 0, 1, 0, \ldots, 1, 0)$.

(b) $(x_1, \ldots, x_{i-1}, 0)(2, 3, 1, 0, 1, 0, \ldots, 1, 0)$.

Using induction and Lemma 4.2 with $\beta_n \geq 0.63b^{n-1}$ as well as Corollary 4.3(ii) we have

$$\begin{aligned}\mu_{2k+1}(s_{2k+1}) &\geq \frac{cb^{j+1}}{8^{2(j+1)}}\frac{\beta_{k-j}}{8^{2k-2j-1}} + \frac{cb}{3}\frac{b^j}{8^{2j+1}}\frac{4b^{k-j-1}}{8^{2(k-j)}}\\ &\geq \frac{cb^k}{8^{2k+1}}\left(0.63 + \frac{4}{3}\right) > \frac{cb}{3}\frac{b^k}{8^{2k+1}}.\end{aligned}$$

This proves the claim of Step 1.

Step 2. To prove (4.1), assume that the statement is true for $2n$ and then use Step 1 to prove the case $2n+1$. Then follow by the same induction method as in Step 1 to prove the case $2(n+1)$. ∎

In the above we see that if $\mathbf{x} = (2, 2, \ldots)$ or $(0, 1, 0, 1, \ldots)$, then the corresponding sum $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j$ has the smallest local dimension $\underline{\alpha}$ and the second largest local dimension $\tilde{\alpha}$, respectively. We will show that $(\underline{\alpha}, \tilde{\alpha})$ is the essential range of the local dimension of μ, i.e., for $\alpha \in (\underline{\alpha}, \tilde{\alpha})$; by suitably arranging the above two patterns of $\mathbf{x}$, we can find an $s \in \operatorname{supp}\mu$ such that $\alpha(s) = \alpha$.

We need a few notations. Let $\{k_j\}_{j=1}^{\infty}$ be a sequence of positive integers, let n_l be the lth partial sum of $\{k_j\}_{j=1}^{\infty}$, and e_l and o_l are the respective sums of the even and odd terms of $\{k_j\}_{j=1}^{l}$. Obviously $n_l = o_l + e_l$.

LEMMA 4.5. *Let* $s = \Sigma_{j=1}^{\infty} 3^{-j} x_j$, *where*

$$\mathbf{x} = (\underbrace{2, \ldots, 2}_{k_1}, \underbrace{2, 0, 1, 0, \ldots, 1, 0}_{k_2}, \underbrace{2, \ldots, 2}_{k_3}, \underbrace{2, 0, 1, 0, \ldots, 1, 0}_{k_4}, \ldots).$$

Then there exists $c, d > 0$ *such that*

$$\frac{c^{[l/2]} 3^{o_l} b^{e_l/2}}{8^{n_l}} \le \mu_{n_l}(s_{n_l}) \le \frac{d^{[l/2]} 3^{o_l} b^{e_l/2}}{8^{n_l}}. \tag{4.2}$$

Proof. We can modify Corollary 4.3(i) for $\bar{s} = \Sigma_{j=1}^{\infty} 3^{-j} \bar{x}_j$ with $\bar{\mathbf{x}} = (2, 0, 1, 0, 1, \ldots)$ to find $c, d > 0$ such that

$$\frac{cb^{n/2}}{8^n} \le \mu_n(\bar{s}_n) \le \frac{db^{n/2}}{8^n}. \tag{4.3}$$

We now use induction to prove (4.2). For $l = 1$, $\mu_{n_1}(s_{n_1}) = (\frac{3}{8})^{o_1}$ and the lemma is trivially true. For $l = 2$, suppose $(y_1, \ldots, y_{k_1}, y_{k_1+1}, \ldots, y_{k_1+k_2})$ is another representation of $s_{k_1+k_2}$ corresponding to $(x_1, \ldots, x_{k_1+k_2}) = (2, \ldots, 2, 2, 0, 1, \ldots, 1, 0)$. We first claim that $y_1 = 2$. Otherwise by Proposition 2.3, we necessarily have

$$(x_1 - y_1, \ldots, x_{k_1+1} - y_{k_1+1}) = (-1, 2, \ldots, 2).$$

Also by the same lemma, $x_{k_1+2} - y_{k_1+2} = 0 - y_{k_1+2} = 2$ or 3, which is impossible. Hence $y_1 = 2$, and the same proof shows that $y_2 = \cdots = y_{k_1} = 2$. Therefore

$$\mu_{n_2}(s_{n_2}) = \left(\frac{3}{8}\right)^{n_1} \mu_{k_2}(\bar{s}_{k_2}),$$

which satisfies (4.2) by (4.3). Suppose (4.2) is true for $l - 1$ where l is odd. By the same argument as the case $l = 2$, we have $y_i = 2$ for all $n_{l-1} + 1 \le j \le n_l$. Hence

$$\mu_{n_l}(s_{n_l}) = \mu_{n_{l-1}}(s_{n_{l-1}}) \left(\frac{3}{8}\right)^{k_l}$$

and

$$\mu_{n_l+1}(s_{n_l+1}) = \mu_{n_{l-1}}(s_{n_{l-1}}) \left(\frac{3}{8}\right)^{k_l} \mu_{k_l+1}(\bar{s}_{k_l+1}).$$

This proves the estimate in (4.2). ∎

THEOREM 4.6. *Let* μ *and* $\underline{\alpha}$, $\tilde{\alpha}$ *be defined as before. Then for any* $\alpha \in (\underline{\alpha}, \tilde{\alpha})$, *there exists* $s \in (0, l)$ *such that* $\alpha(s) = \alpha$.

Proof. Recall that $\underline{\alpha} = \frac{3\log 2}{\log 3} - 1$, $\tilde{\alpha} = \frac{3\log 2}{\log 3} - \frac{\log b}{2\log 3}$. Then for $\alpha \in (\underline{\alpha}, \tilde{\alpha})$ we can write $\alpha = \theta\underline{\alpha} + (1-\theta)\tilde{\alpha}$ for some $0 < \theta < 1$. Let

$$k_j = \begin{cases} 2j & \text{if } j \text{ is odd} \\ 2\left[\frac{j(1-\theta)}{\theta}\right] & \text{if } j \text{ is even.} \end{cases}$$

Let $s = \sum_{j=1}^{\infty} 3^{-j} x_j$ be the form as in Lemma 4.5 with k_j so defined. Then

$$\lim_{l\to\infty} \frac{o_l}{n_l} = \theta, \qquad \lim_{\ell\to\infty} \frac{\ell}{n_l} = 0, \qquad \text{and} \qquad \lim_{l\to\infty} \frac{n_{l-1}}{n_l} = 1.$$

By (4.2) and a direct calculation we have

$$\alpha(s) = \lim_{n\to\infty} \left| \frac{\log \mu_n(s_n)}{n \log 3} \right| = \alpha.$$

∎

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