ASIAN J. MATH. © 2000 International Press
Vol. 4, No. 3, pp. 527–552, September 2000 003

ITERATED FUNCTION SYSTEMS WITH OVERLAPS*

AI-HUA FAN†, KA-SING LAU‡, AND SZE-MAN NGAI§

Abstract. In general it is difficult to study the multifractal structure of a self-similar measure when the associated iterated function system does not satisfy the open set condition. In this paper we will give two methods to deal with the overlapping situation. For the first method we make use of a transition matrix to calculate the L^p-scaling spectrum $\tau(p)$ of the measure. The second method depends on the Fourier transformation and the Ruelle operator; we use it to calculate the Sobolev exponent of the measure. We apply these two methods to study the m-th convolution of the Cantor measure, and also an interesting construction investigated recently by Kenyon [K] and Rao and Wen [RW]: $S_0(x) = \frac{1}{3}x,\ S_1(x) = \frac{1}{3}x + \frac{\lambda}{3},\ S_2(x) = \frac{1}{3}x + \frac{2}{3}$, with $0 < \lambda < 1,\ \lambda \in \mathbb{Q}$.

1. Introduction. Let μ be a bounded positive Borel measure on $\mathbb{R}^d$ with compact support. For $p > 0$, the *L^p-scaling spectrum (*or *L^p-scaling exponent)* $\tau(p)$ of μ is defined as

$$\tau(p) = \lim_{h\to 0^+} \frac{\ln \sum_i \mu(Q_i(h))^p}{\ln h}, \tag{1.1}$$

where $Q_i(h) = [n_1h,\ (n_1{+}1)h)\times\cdots\times[n_dh,\ (n_d{+}1)h),\ (n_1,\ldots,n_d) \in \mathbb{Z}^d$ is an h-mesh cube in $\mathbb{R}^d$ and the sum is taken over all such cubes which intersect the support of μ. (A more general definition of $\tau(p)$ which includes negative values of p can be found in [LN1], [O], [R].) The function $\tau(p)$ plays a central role in the theory of multifractal measures. It is well known that if μ is the self-similar measure defined by a family of contractive similitudes $\{S_j\}_{j=0}^m$ which satisfies the *open set condition* [Hu], then there is an explicit formula to calculate $\tau(p)$ ([CM], [O]). Moreover, the Legendre transformation (concave conjugate) of $\tau(p)$ (i.e., $\tau^*(\alpha) := \inf\{q\alpha - \tau(q) : q \in \mathbb{R}\}$) equals the Hausdorff dimension of the set

$$K(\alpha) = \Big\{x \in \mathrm{supp}(\mu) : \lim_{h\to 0^+} \frac{\ln \mu(B_h(x))}{\ln h} = \alpha\Big\},$$

where $B_h(x)$ denotes the h-ball centered at x, and $\mathrm{supp}(\mu)$ denotes the support of μ. The quantity $\lim\limits_{h\to 0^+} \dfrac{\ln \mu(B_h(x))}{\ln h}$ is known as the *local dimension* of μ at x, the Hausdorff dimension of $K(\alpha)$ as a function of α is called the *dimension* of μ, and the relationship between $\tau(p)$ and the dimension is the well-known *multifractal formalism* (see e.g., [F], [CM]). In general it is difficult to handle $K(\alpha)$ theoretically or computationally. The scaling spectrum $\tau(p)$ and the multifractal formalism hence serve as an important alternative to study the multifractal structure.

Following the terminology of Barnsley [B], we call the above family of contractive similitudes $\{S_j\}_{j=0}^m$ an *iterated function system* (IFS). If the family does not satisfy the open set condition, it is much harder to obtain the scaling exponent $\tau(p)$ and

*Received August 14, 1999; accepted for publication February 9, 2000.

†Département de Mathématiques, Université de Picardie, 33, rue Saint Leu, 80039 Amiens, France (ai-hua.fan@u-picardie.fr). Research supported by the Institute of Mathematical Sciences of the Chinese University of Hong Kong.

‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong (kslau@math.cuhk.edu.hk). Research partially supported by an RGC Grant from CUHK.

§School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA (ngai@math.gatech.edu). Research supported by a postdoctoral fellowship from the Chinese University of Hong Kong.

it is not known whether the multifractal formalism will hold in general [LN1]. For example, for the standard Cantor measure μ, the IFS

$$S_0 x = \frac{1}{3}x, \quad S_1 x = \frac{1}{3}x + \frac{2}{3}$$

satisfies the open set condition and it is well known that the L^p-spectrum of μ is $(p-1)\frac{\ln 2}{\ln 3}$. The m-time convolution $\nu = \mu * \cdots * \mu$ satisfies the following self-similar relation

$$\nu = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}\nu \circ S_j^{-1},$$

with $S_j(x) = \frac{1}{3}x + \frac{2}{3}j$, $0 \le j \le m$ (see (3.2)). Note that this new family $\{S_j\}_{j=0}^m$ does not satisfy the open set condition when $m \ge 3$ and nothing is known about the scaling spectrum and the local structure of ν. Of course it is easy to see that the support of ν is an interval and ν is singular (because $\hat{\mu}(\xi) \not\to 0$ as $\xi \to \infty$ [JW], and the same holds for $\hat{\nu}(\xi)(= \hat{\mu}(\xi)^m)$). In [St1] and [St2], Strichartz made some preliminary study of such convolution (under a slightly more general setting) and estimated the corresponding scaling exponent through the Fourier transform. However in his work, he had to assume the open set condition on the family of similitudes defining the convolution.

Another interesting study of this type of IFS has recently been carried out independently by Kenyon [K] and Rao and Wen [RW]. Let

$$S_0(x) = \frac{1}{3}x, \quad S_1(x) = \frac{1}{3}x + \frac{\lambda}{3}, \quad S_2(x) = \frac{1}{3}x + \frac{2}{3},$$

where $0 < \lambda < 1$ and let F_λ be the associated *self-similar set* (or *attractor*) of the IFS. They showed that for $\lambda = a/b$ a rational number, if $a \not\equiv b$ (mod 3), then the open set condition fails and F_λ has Lebesgue measure zero. Surprisingly if $a \equiv b$ (mod 3), then the open set condition is satisfied and F_λ has nonempty interior. In the first case, nothing is known about the multifractal structure of the corresponding self-similar measure μ (with weight $1/3$ on each map). In the second case, nothing is known about the smoothness of the density function. We call such μ a λ-*Cantor measure*, using the terminology in [RW]. In the first case we would like to know the L^p-scaling spectrum of μ. In the second case μ is absolutely continuous and we would like to know the Sobolev exponent of the corresponding density function.

In this paper we will use two approaches to investigate the questions raised above. For the first approach we need to make use of a new concept of separation of an IFS, which extends the open set condition. We say that a family of similitudes

$$S_j(x) = \rho x + b_j, \quad 0 < \rho < 1, \quad b_j \in \mathbb{R}, \quad j = 0,\ 1, \dots, m$$

has the *weak separation property* (WSP) if there exists $\delta > 0$ such that for $J = (j_1, \dots, j_n)$, $J' = (j'_1, \dots, j'_n)$,

$$\text{(1.2)} \qquad \text{either} \quad S_J(0) = S_{J'}(0) \quad \text{or} \quad |S_J(0) - S_{J'}(0)| \ge \delta\rho^n.$$

(S_J denotes the composition $S_{j_1} \circ \cdots \circ S_{j_n}$.) The WSP defined under the present setting is equivalent to the more general definition introduced in [LN1] where the contraction ratios ρ_j can be different for different S_j and the domain of S_j is $\mathbb{R}^d$. Under the general form of the weak separation condition and the assumption that $\tau^*(\alpha)$ is strictly concave, the multifractal formalism described in the first paragraph is proved to be valid [LN1]. By using the maximal eigenvalue of some finite nonnegative transition matrix, we can calculate $\tau(p)$ for p equal to a positive integer. This method has already been developed in [LN3] and it applies not only to the above two cases with $\rho = 1/3$, but also to the more general case when ρ^{-1} is a *P.V. number* (i.e., an algebraic integer whose algebraic conjugates have moduli less than 1 [S]. Positive integers ≥ 2 and the golden ratio $(\sqrt{5}+1)/2$ are examples of P.V. numbers.) P.V. numbers play an important role in obtaining the finite transition matrix.

Our second method is to use Fourier transformation. Recall that for a self-similar measure μ generated by contractive similitudes $\{S_j\}_{j=0}^m$ with contraction ratio ρ, we have

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} P(\rho^n \xi), \tag{1.3}$$

for some trigonometric polynomial P. For a locally integrable function G on $\mathbb{R}$, we define

$$\beta(q) = \sup\Big\{s : \int_{-\infty}^{\infty} (1+|\xi|^q)^s |G(\xi)|^q \, d\xi < \infty\Big\}, \quad q \in \mathbb{R}. \tag{1.4}$$

Note that if $G = \hat{f}$, then $\beta(2)$ is the Sobolev exponent of f. In the measure case we have $f = \dfrac{d\mu}{dx}$ (in the distributional sense if μ is singular).

We are interested in calculating $\beta(q)$ for the λ-Cantor measures (the case of convolution of the Cantor measure can be handled similarly). The technique is to make use of a setup in dynamical systems (see [Bo], [Ru], [FL], [H]). Let g be a nonnegative Hölder continuous 1-periodic function with $g(0) = 1$. Let L_g be defined on the space of continuous functions f in $C[0,1]$ by

$$L_g f(x) = g(\frac{x}{3})f(\frac{x}{3}) + g(\frac{x}{3}+\frac{1}{3})f(\frac{x}{3}+\frac{1}{3}) + g(\frac{x}{3}+\frac{2}{3})f(\frac{x}{3}+\frac{2}{3}).$$

L_g is called the *Ruelle operator*. Let ρ_g denote the spectral radius of L_g. By using $g(\xi) = |P(\xi)|$ as in (1.3), we obtain a formula relating ρ_g^q and $\beta(q)$ (Theorem 5.4). To calculate ρ_g (or ρ_{g^q}), we make use of an observation of Hervé [H]: $\rho_g = \lim_{n\to\infty} \|L_g^n 1\|^{1/n}$ (use the supremum norm), and the value will be the same if we restrict the operator to an invariant subspace containing 1. In the case of λ-Cantor measures, we can find such a finite dimensional subspace and ρ_g can be calculated.

We organize the paper as follows. In Section 2, we present an algorithm to calculate $\tau(p)$ for p a positive integer. This method is modified from the one used in [LN3]. We use this algorithm to study convolutions of the Cantor measure in Section 3 (Theorems 3.2 and 3.3), and the λ-Cantor measures in Section 4. In Section 5, we consider the Fourier transform method. To obtain $\beta(q)$ it is more convenient to replace the integral in (1.4) by $\int_{|\xi|<T} |\xi|^{qs}|G(\xi)|^q \, d\xi$. A general theory of this has been developed in [FL]. Again we have modified it into the present setting. We implement the method by calculating $\beta(2)$ for the λ-Cantor measures. In the case the measure is singular, the values match well with the $\tau(2)$ calculated by using the first method; this is justified by Theorem 5.1.

2. The basic theorem to calculate $\tau(p)$. We first observe that on $\mathbb{R}$, the expression $\sum_i \mu(Q_i(h))^q$ is the Riemann sum of the integral $\frac{1}{h}\int_{-\infty}^{\infty} |\mu(B_h(t))|^p\, dt$, where $B_h(t) := [t-h,\ t+h)$. Hence we can modify the definition of $\tau(p)$ in (1.1) into

$$\begin{aligned}\tau(p) &= \varliminf_{h\to 0^+}\Big(\ln\Big(\int_{-\infty}^{\infty}|\mu(B_h(t))|^p\,dt\Big)/\ln h\Big) - 1\\ &= \inf\Big\{\alpha : \varlimsup_{h\to 0^+}\frac{1}{h^{1+\alpha}}\int_{-\infty}^{\infty}|\mu(B_h(t))|^p\,dt > 0\Big\}\end{aligned}$$

(see [L1], [St3]). In this section we will show that for the class of self-similar measures under consideration, we can find $\alpha = \tau(p)$ such that $0 < \varlimsup_{h\to 0^+}\frac{1}{h^{1+\alpha}}\int_{-\infty}^{\infty}|\mu(B_h(t))|^p\,dt$ $< \infty$. The basic idea has been developed in [L2] and [LN3]. Here we will modify it to fit our more general case. Some key ideas are included and the details can be found in [LN3].

Let

$$S_j(x) = \rho x + b_j, \quad j = 0, 1, \dots, m.$$

For convenience we can assume that $b_j \ge 0$ for all j with $b_0 = 0$. Let

$$\mu = \sum_{j=0}^{m} w_j \mu \circ S_j^{-1}, \tag{2.1}$$

where $\{w_j\}_{j=0}^m$ is a set of probability weights, i.e., $w_j \ge 0$ and $\sum_{j=0}^m w_j = 1$. Then $\operatorname{supp}(\mu)$ is contained in $[0, C]$, where $C = b_{\max}/(1-\rho)$ and $b_{\max} = \max\{b_j : j = 1, \dots, m\}$.

Let p be a fixed positive integer. For $\alpha \ge 0$, $h > 0$, and $\boldsymbol{s} = (s_1, \dots, s_p)$, let

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}}\int_{-\infty}^{\infty}\mu\big(B_h(t+s_1)\big)\cdots\mu\big(B_h(t+s_p)\big)\,dt. \tag{2.2}$$

Note that $\Phi^{(\alpha)}(h) := \Phi_{\boldsymbol{0}}^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}}\int_{-\infty}^{\infty}\mu\big(B_h(t)\big)^p\,dt$. For our purpose one simple way to interpret the $\boldsymbol{s}$ in (2.2) geometrically is to think of it in the following form

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}}\int_{\gamma_{\boldsymbol{s}}}\mu\big(B_h(x_1)\big)\cdots\mu\big(B_h(x_p)\big)\,d\gamma,$$

where $\gamma_{\boldsymbol{s}}$ is the line with direction vector $(1, \dots, 1)$ passing through the point $\boldsymbol{s}$ and $\int_{\gamma_{\boldsymbol{s}}}$ denotes the line integral along $\gamma_{\boldsymbol{s}}$, i.e., we regard $\boldsymbol{s}$ as $\gamma_{\boldsymbol{s}}$ (see the diagram in Figure 2.1). It is easy to see that if $\boldsymbol{s}' = \boldsymbol{s} + c(1, \dots, 1)$, then

$$\Phi_{\boldsymbol{s}'}^{(\alpha)}(h) = \Phi_{\boldsymbol{s}}^{(\alpha)}(h). \tag{2.3}$$

Also, by substituting (2.1) into (2.2) and making use of a change of variables, we have

$$\begin{aligned}\Phi_{\boldsymbol{s}}^{(\alpha)}(h) &= \frac{1}{h^{1+\alpha}}\int_{-\infty}^{\infty}\prod_{i=1}^{p}\Big(\sum_{k_i=0}^{m} w_{k_i}\mu\Big(B_{\frac{h}{\rho}}\big(\frac{t}{\rho}+\frac{s_i}{\rho}-\frac{b_{k_i}}{\rho}\big)\Big)\Big)\,dt\\ &= \frac{1}{\rho^\alpha(\frac{h}{\rho})^{1+\alpha}}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}\prod_{i=1}^{p} w_{k_i}\mu\Big(B_{\frac{h}{\rho}}\big(t+\frac{s_i}{\rho}-\frac{b_{k_i}}{\rho}\big)\Big)\,dt,\end{aligned}$$

with $\boldsymbol{k} = (k_1, \dots, k_p)$ and $k_j \in \{0, 1, \dots, m\}$. Hence we have the following identity

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{\rho^\alpha} \sum_{\boldsymbol{k}} w_{k_1} \cdots w_{k_p} \Phi_{\boldsymbol{s}^{\boldsymbol{k}}}^{(\alpha)}(\frac{h}{\rho}) := \frac{1}{\rho^\alpha} \sum_{\boldsymbol{k}} w_{\boldsymbol{k}} \Phi_{\boldsymbol{s}^{\boldsymbol{k}}}^{(\alpha)}(\frac{h}{\rho}), \tag{2.4}$$

where

$$\boldsymbol{s}^{\boldsymbol{k}} = \rho^{-1}(\boldsymbol{s} - \boldsymbol{b}_{\boldsymbol{k}}), \quad \boldsymbol{b}_{\boldsymbol{k}} = (b_{k_1}, \dots, b_{k_p}), \quad \text{and} \quad w_{\boldsymbol{k}} = w_{k_1} \cdots w_{k_p}. \tag{2.5}$$

We define a set of states $\mathcal{S}$ inductively as follows: Let $\boldsymbol{s}_0 = \mathbf{0} = (0, \dots, 0) \in \mathbb{R}^p$ be the initial state. Suppose all states on level $n-1$ have been defined. Then level n consists of all possible states of the form $\boldsymbol{s}^{\boldsymbol{k}}$, where $\boldsymbol{s}$ is a state on level $n-1$, $\boldsymbol{k} = (k_1, \dots, k_p)$, with $k_j \in \{0, 1, \dots, m\}$.

Next, we define a transition matrix on such states. For a fixed integer $p \geq 2$ and $\boldsymbol{s} = (s_1, \dots, s_p)$, we let

$$T\boldsymbol{s} = \sum_{\boldsymbol{k}}{}' w_{\boldsymbol{k}} \cdot \boldsymbol{s}^{\boldsymbol{k}}, \tag{2.6}$$

(see Figure 2.1) and let $\langle \mathcal{S} \rangle$ denote the linear space spanned by $\mathcal{S}$. By regarding $\boldsymbol{s}$ as a word (or as the line $\gamma_{\boldsymbol{s}}$) and adopting the convention that

$$\Phi_{a\cdot\boldsymbol{s}_1 + b\cdot\boldsymbol{s}_2}^{(\alpha)}(h) = a\Phi_{\boldsymbol{s}_1}^{(\alpha)}(h) + b\Phi_{\boldsymbol{s}_2}^{(\alpha)}(h),$$

(2.4) reduces to

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{\rho^\alpha} \Phi_{T\boldsymbol{s}}^{(\alpha)}(\frac{h}{\rho}). \tag{2.7}$$

Remark. Note that in (2.5) $\boldsymbol{s}$ and $\boldsymbol{s}^{\boldsymbol{k}}$ are regarded as vectors, while in (2.6) they are regarded as "words" of the vector space $\langle \mathcal{S} \rangle$. To avoid confusion we use the notation $\sum_i' c_i \cdot \boldsymbol{s}_i$ to emphasize that the linear combination is taken in $\langle \mathcal{S} \rangle$, not in $\mathbb{R}^d$. We also write $r \sum_i' c_i \cdot \boldsymbol{s}_i$, $r \in \mathbb{R}$, to mean $\sum_i'(rc_i) \cdot \boldsymbol{s}_i$, and $r \sum_i' \boldsymbol{s}_i$ to mean $\sum_i' r \cdot \boldsymbol{s}_i$.

On the set of states defined above, we identify $\boldsymbol{s}'$ with $\boldsymbol{s}$ if and only if $\boldsymbol{s}' = \boldsymbol{s} + c(1, \dots, 1)$ for some number c (see (2.3)). We denote the quotient set of states under this identification by the same notation $\mathcal{S}$. It is easy to extend T to a linear map $T : \langle \mathcal{S} \rangle \to \langle \mathcal{S} \rangle$.

Proposition 2.1. *Let $C = b_{\max}/(1-\rho)$ and let*

$$\mathcal{S}_1 = \{\boldsymbol{s} \in \mathcal{S} : |s_i - s_j| \leq C \text{ for all } 1 \leq i, j \leq p\}.$$

Then T is invariant on $\langle \mathcal{S} \backslash \mathcal{S}_1 \rangle$.

Proof. Let $\boldsymbol{s} \in \mathcal{S} \backslash \mathcal{S}_1$. Then there exist $1 \leq i_0, j_0 \leq p$ such that $|s_{i_0} - s_{j_0}| > C$. Note that

$$T\boldsymbol{s} = \sum_{\boldsymbol{k}}{}' w_{\boldsymbol{k}} \cdot \boldsymbol{s}^{\boldsymbol{k}} = \sum_{\boldsymbol{k}}{}' w_{\boldsymbol{k}} \cdot \big(\rho^{-1}(\boldsymbol{s} - \boldsymbol{b}_{\boldsymbol{k}})\big).$$

For each $\boldsymbol{k}$, we have

$$\big|\rho^{-1}(s_{i_0} - b_{k_{i_0}}) - \rho^{-1}(s_{j_0} - b_{k_{j_0}})\big| > \rho^{-1}(C - b_{\max}) = \rho^{-1}\big(C - (1-\rho)C\big) = C.$$

Hence $\boldsymbol{s}^{\boldsymbol{k}} \in \mathcal{S}\backslash\mathcal{S}_1$ and the result follows. □

In the case that $\mathcal{S}_1$ is a finite set,

$$T = \begin{bmatrix} T_1 & 0 \\ Q & T_2 \end{bmatrix}$$

where T_1 corresponds to the states $\mathcal{S}_1$. T_1 is a sub-Markov matrix (since the sum of each column of T is 1 so that the sum of each column of T_1 is less than or equal to 1) and it is the basic matrix we use to calculate $\tau(p)$ for μ. The following proposition provides a sufficient condition for $\{S_j\}_{j=0}^m$ to have the WSP and $\mathcal{S}_1$ to be finite.

PROPOSITION 2.2. *Let $a \in \mathbb{R}$, $0 < \rho < 1$, and let $S_j(x) = \rho x + ar_j$, where $0 \le j \le m$ and r_j are rational numbers. If $\beta = \rho^{-1}$ is a P.V. number, then $\{S_j\}_{j=0}^m$ has the WSP and $\mathcal{S}_1$ is a finite set.*

Proof. Since the r_j's are rational, we can write $r_j = \tilde{r}_j/r$ for $0 \le j \le m$, where $r,\ \tilde{r}_j \in \mathbb{Z}$. Then for any $J = (j_1, \dots, j_n)$, $1 \le j_k \le m$,

$$S_J(0) = \frac{a}{\rho}\sum_{k=1}^{n} r_{j_k}\rho^k = \rho^{n-1}\frac{a}{r}\Big(\sum_{k=1}^{n}\tilde{r}_{j_k}\beta^{n-k}\Big).$$

Hence for $J' = (j'_1, \dots, j'_n)$,

$$|S_J(0) - S_{J'}(0)| = \rho^{n-1}\frac{a}{r}\Big|\sum_{k=1}^{n}(\tilde{r}_{j_k} - \tilde{r}_{j'_k})\beta^{n-k}\Big|.$$

Since β is a P.V. number, a lemma of Garsia ([G, Lemma 1.51]) implies that there exists $\delta > 0$ (independent of n) such that if $\sum_{k=1}^{n}(\tilde{r}_{j_k} - \tilde{r}_{j'_k})\beta^{n-k} \ne 0$, then

$$\Big|\sum_{k=1}^{n}(\tilde{r}_{j_k} - \tilde{r}_{j'_k})\beta^{n-k}\Big| \ge \delta.$$

Hence $|S_J(0) - S_{J'}(0)| \ge \dfrac{a\delta}{\rho r}\cdot\rho^n$, and $\{S_j\}_{j=0}^m$ has the WSP.

To show that $\mathcal{S}_1$ is finite, we observe that for any $\boldsymbol{s}$ in the n-th iteration as defined in (2.5), starting from $\boldsymbol{s}_0$,

$$s_i = -\big(\rho^{-(n-1)}b_{j_1} + \rho^{-(n-2)}b_{j_2} + \cdots + \rho^{-1}b_{j_{n-1}} + b_{j_n}\big) = -\rho^{-(n-1)}S_J(0).$$

Let $d_{ij} = s_i - s_j$, where $s_j = -\rho^{-(n-1)}S_{J'}(0)$ is any other such coordinate. Then $\boldsymbol{s} \in \mathcal{S}_1$ if and only if $|d_{ij}| \le C$ for all $1 \le i, j \le p$. It suffices to show that the distinct d_{ij}'s are separated by at least some fixed positive constant. Note that

$$|d_{ij} - \tilde{d}_{ij}| = \rho^{-(n-1)}\big|(S_J(0) - S_{J'}(0)) - (\tilde{S}_J(0) - \tilde{S}_{J'}(0))\big|.$$

As in the proof of the WSP, the right hand side of this expression can be written as

$$\frac{a}{r}\Big|\sum_{k=1}^{n}\eta_k\beta^{n-k}\Big|,$$

where η_k belongs to some finite set of integers (independent of $d_{ij}, \tilde{d}_{ij}$). The same lemma of Garsia implies that there exists some $\delta' > 0$ such that either $d_{ij} = \tilde{d}_{ij}$ or $|d_{ij} - \tilde{d}_{ij}| > \delta'$. This completes the proof. □

Let μ be defined as in (2.1). It follows from a similar proof as in [L2, Proposition 3.1] (also [LN3, Proposition 3.2]) that if $\Phi_{\boldsymbol{s}}^{(\alpha)}(h) > 0$ for all $h > 0$ then $\boldsymbol{s} \in \mathcal{S}_1$; the converse holds if some representation $\boldsymbol{s}'$ of $\boldsymbol{s}$ satisfies $\boldsymbol{s}' \in \underbrace{\operatorname{supp}(\mu) \times \cdots \times \operatorname{supp}(\mu)}_{p}$ (see Figure 2.1). By using this fact, we see that if $\mathcal{S}_1$ is finite, then there exists $h_0 > 0$ such that for all $0 < h < h_0$, (2.7) holds with T_1 replacing T, i.e.,

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{\rho^\alpha} \Phi_{T_1 \boldsymbol{s}}^{(\alpha)}(\frac{h}{\rho}).$$

If λ is the maximal eigenvalue of T_1 with maximal eigenvector $\boldsymbol{u} = \sum' a_i \cdot \boldsymbol{s}_i$, and if α satisfies $\rho^\alpha = \lambda$, then the above identity becomes

$$\Phi_{\boldsymbol{u}}^{(\alpha)}(h) = \Phi_{\boldsymbol{u}}^{(\alpha)}(\frac{h}{\rho}), \tag{2.8}$$

and it follows that $0 < \overline{\lim}_{h \to 0^+} \Phi_{\boldsymbol{u}}^{(\alpha)}(h) < \infty$. From this we can show (see [L2, Theorem 4.2] for details)

THEOREM 2.3. *Let $p \geq 2$ be a positive integer and let μ be the self-similar measure defined by $\{S_j\}_{j=0}^m$ with a set of probability weights $\{w_j\}_{j=0}^m$. Suppose $\mathcal{S}_1$ is a finite set. Then $\tau(p) = \ln \lambda / \ln \rho$, where λ is the maximal eigenvalue of T_1.*

Combining Proposition 2.2 and Theorem 2.3 we have

THEOREM 2.4. *Let $a \in \mathbb{R}$, $0 < \rho < 1$, and let $S_j(x) = \rho x + a r_j$, where $0 \leq j \leq m$ and r_j are rational numbers. Suppose $\beta = \rho^{-1}$ is a P.V. number. Then $\tau(p) = \ln \lambda / \ln \rho$, where λ is the maximal eigenvalue of T_1.*

In many calculations, it is more convenient to reduce the size of $\mathcal{S}_1$ and T_1: For $\boldsymbol{s} = (s_1, \ldots, s_p) \in \mathcal{S}_1$, we let $\boldsymbol{s}_\sigma$ be a decreasing rearrangement of $\boldsymbol{s}$ (i.e., $s_{\sigma(1)} \geq s_{\sigma(2)} \geq \cdots \geq s_{\sigma(p)}$). It follows directly from definition that

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \Phi_{\boldsymbol{s}_\sigma}^{(\alpha)}(h).$$

Let $\mathcal{S}_1^\sigma$ be the set of all such $\boldsymbol{s}_\sigma$. (See Figure 2.2 for the prismatic region; we take $\boldsymbol{s}_\sigma$ to be the representative on the triangular base.) We summarize the above into the following algorithm to calculate λ when $\mathcal{S}_1$ is finite:

Step (I). The set $\mathcal{S}_1^\sigma$: For $\boldsymbol{s} \in \mathcal{S}_1$, we can use (2.3) to choose the representation with the last coordinate equal to 0, i.e., all representations in $\mathcal{S}_1$ are of the form $(\boldsymbol{t}, 0) \in \mathcal{S}_1$ (the shaded region in Figure 2.1). Starting from $\mathbf{0}$, suppose we have chosen $(\boldsymbol{t}, 0) \in \mathcal{S}_1^\sigma$ (the triangular region in Figure 2.2) in step $(n-1)$. Let $\boldsymbol{s}$ be a state in step n, i.e., there exists $\boldsymbol{k}$ such that

$$\boldsymbol{s} = (\boldsymbol{t}, 0)^{\boldsymbol{k}} = \rho^{-1}\big((\boldsymbol{t}, 0) - \boldsymbol{b}^{\boldsymbol{k}}\big)$$

as in (2.5). Rearrange $\boldsymbol{s}$ to $\boldsymbol{s}_\sigma$ so that $s_{\sigma(1)} \geq \cdots \geq s_{\sigma(p)}$ and let

$$(\boldsymbol{t}', 0) = (s_{\sigma(1)} - s_{\sigma(p)}, \ldots, s_{\sigma(p-1)} - s_{\sigma(p)}, 0).$$

Do this for all elements $(\boldsymbol{t}, 0) \in \mathcal{S}_1^\sigma$ in step $(n-1)$. If there is no new $(\boldsymbol{t}', 0)$ we stop the process; otherwise we continue onto the next step. This process will stop after a finite number of iterations because $\mathcal{S}_1$ is a finite set. (The set $\mathcal{S}_1^\sigma$ is defined by the shaded region in Figure 2.2.)

Step (II). The canonical matrix T_1^σ on $\mathcal{S}_1^\sigma$: For each $(\boldsymbol{t},0) \in \mathcal{S}_1^\sigma$, the entry corresponding to $(\boldsymbol{t}',0) \in \mathcal{S}_1^\sigma$ is given by

$$\sum_{\boldsymbol{k}} \Big\{ w_{\boldsymbol{k}} : (\boldsymbol{t},0)^{\boldsymbol{k}} \text{ equals } (\boldsymbol{t}',0) \text{ after the rearrangement in Step (I)} \Big\}.$$

The matrix T_1^σ so constructed has the same maximal eigenvalue λ as T_1 [LN3, Proposition 2.3].

Remark 1. The case $p=2$ was first considered in [L2]. The notation is much simpler: We omit the last coordinate and consider $\mathcal{S}_1$ to be obtained directly from the iterations

$$t_n = \rho^{-1}(t_{n-1}+c_n), \quad |t_n| \le C,$$

where c_n is of the form $b_i - b_j$ (see (2.5) and Step (I)). The corresponding $\mathcal{S}_1^\sigma$ is obtained by replacing t_n with $|t_n|$.

Remark 2. Let

$$\partial\mathcal{S}_1^\sigma = \big\{ \boldsymbol{s} \in \mathcal{S}_1^\sigma : s_1 = C \big\}.$$

The region defining $\partial\mathcal{S}_1^\sigma$ is part of the boundary of the area defining $\mathcal{S}_1^\sigma$ (see Figure 2.2). We claim that $\boldsymbol{s} \in \partial\mathcal{S}_1^\sigma$ implies the rearrangement of $\boldsymbol{s}^{\boldsymbol{k}}$ does not belong to $\mathcal{S}_1^\sigma \backslash \partial\mathcal{S}_1^\sigma$. To prove this, we let $\boldsymbol{s} \in \partial\mathcal{S}_1^\sigma$. Then

$$\boldsymbol{s}^{\boldsymbol{k}} = \big(\rho^{-1}(C - b_{j_1}), \ldots, \rho^{-1}(s_{p-1} - b_{j_{p-1}}), \rho^{-1}(-b_{j_p})\big).$$

If $b_{j_1} < b_{\max}$, then

$$\rho^{-1}(C - b_{j_1}) = \rho^{-1}\big(\frac{b_{\max}}{1-\rho} - b_{j_1}\big) > \rho^{-1}\big(\frac{b_{\max}}{1-\rho} - b_{\max}\big) = C,$$

which implies that $\boldsymbol{s}^{\boldsymbol{k}} \notin \mathcal{S}_1$. Hence for $\boldsymbol{s}^{\boldsymbol{k}}$ to belong to $\mathcal{S}_1$, b_{j_1} must equal $b_{\max}$, and the rearrangement of $\boldsymbol{s}^{\boldsymbol{k}}$ will belong to $\partial\mathcal{S}_1^\sigma$. This proves the claim.

By using this and a similar proof as in [LN3, Proposition 4.2], we can show that the maximal eigenvalue of T_1^σ on $\langle \mathcal{S}_1^\sigma \rangle$ is equal to the maximal eigenvalue of the restriction of T_1^σ on $\langle \mathcal{S}_1^\sigma \backslash \partial\mathcal{S}_1^\sigma \rangle$. We can hence reduce the size of $\mathcal{S}_1^\sigma$ by omitting all those $\boldsymbol{s} \in \mathcal{S}_1^\sigma$ with $s_1 = C$.

Notation. We denote by $\mathcal{S}^\sigma$ the set $\mathcal{S}_1^\sigma \backslash \partial\mathcal{S}_1^\sigma$, and by T^σ the restriction of T_1^σon $\langle \mathcal{S}^\sigma \rangle$.

3. Convolution of the Cantor measure. Let $0 < \rho < 1$ and let $\mu = \mu_\rho$ be the self-similar measure defined by the similitudes

$$\psi_0(x) = \rho x, \quad \psi_1(x) = \rho x + (1-\rho), \tag{3.1}$$

with probabilities w_0 and w_1 respectively.

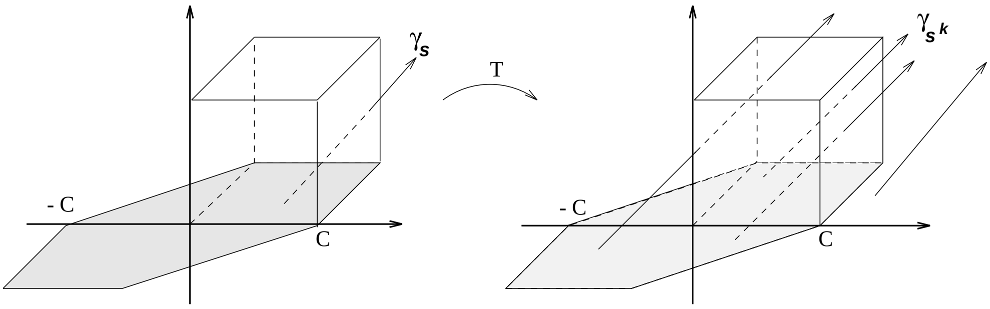

FIG. 2.1. *Diagram showing that the action of T on $\boldsymbol{s}$ (represented by the line $\gamma_{\boldsymbol{s}}$) produces a linear combination of the $\boldsymbol{s}^{\boldsymbol{k}}$ (also represented by lines).*

FIG. 2.2. *Diagram showing the region defining $\mathcal{S}_1^\sigma$. The representatives $\boldsymbol{s}_\sigma = (\boldsymbol{t}, 0)$ lie on the shaded triangular base and ∂S_1^σ is defined by the line segment joining the points $(C, 0)$ and (C, C).*

PROPOSITION 3.1. *Let $\nu = \mu^{*m} := \mu * \cdots * \mu$ (m times). Then ν is the unique self-similar measure defined by*

$$\nu = \sum_{j=0}^{m} \binom{m}{j} w_0^{m-j} w_1^j \nu \circ S_j^{-1}, \tag{3.2}$$

where

$$S_j(x) = \rho x + (1-\rho)j, \quad \text{for } 0 \le j \le m.$$

Moreover supp$(\nu) \subseteq [0, m]$, and supp$(\nu) = [0, m]$ if and only if $\rho \ge \frac{1}{m+1}$.

Proof. For $\nu = \mu * \mu$, a direct calculation shows that

$$\begin{aligned}\nu &= (w_0 \mu \circ \psi_0^{-1} + w_1 \mu \circ \psi_1^{-1}) * (w_0 \mu \circ \psi_0^{-1} + w_1 \mu \circ \psi_1^{-1}) \\ &= w_0^2 \nu \circ S_0^{-1} + 2 w_0 w_1 \nu \circ S_1^{-1} + w_1^2 \nu \circ S_2^{-1}.\end{aligned}$$

The case for $\nu = \mu^{*m}$ follows by induction. For $0 \le j \le m$,

$$S_j[0, m] = [(1-\rho)j,\ m\rho + (1-\rho)j] \subseteq [0, m].$$

It follows that if $\rho < \frac{1}{m+1}$, then $\{S_j[0, m]\}_{j=0}^m$ are nonoverlapping intervals and supp$(\nu) \subseteq \bigcup_{j=0}^m S_j[0, m]$. If $\rho \ge \frac{1}{m+1}$, then $S_j[0, m]$ and $S_{j+1}[0, m]$ have nonvoid intersection and hence $[0, m] = \bigcup_{j=0}^m S_j[0, m]$. $\square$

In view of the proof, we see that $\{S_j\}_{j=0}^m$ satisfies the open set condition if $\rho \le \frac{1}{m+1}$ and $\tau(p)$ for the measure ν can be calculated by an explicit formula ([CM], [LW], [St3]). In the case μ is the standard Cantor measure, then $\nu = \mu * \mu$ is defined by $S_j(x) = \frac{1}{3}x + \frac{2}{3}j$, $j = 0, 1, 2$, and the open set condition is satisfied (an open set is $(0, 2)$). It follows that $\tau(p)$ is given by

$$\tau(p) = \frac{\ln\big(2(\frac{1}{4})^p + (\frac{1}{2})^p\big)}{-\ln 3}, \quad -\infty < p < \infty. \tag{3.3}$$

(See Figure 3.1.)

In the following we will let ν be the m-th convolution of the standard Cantor measure with $m \ge 3$. The family $\{S_j\}_{j=0}^m$ has the WSP (Proposition 2.2). For each p, by using (3.1) and the definition of T in (2.6), we get

$$T(\boldsymbol{s}) = \frac{1}{2^{mp}} \sum_{\boldsymbol{k}}{}' \binom{m}{k_1} \cdots \binom{m}{k_p} \cdot \boldsymbol{s}^{\boldsymbol{k}}, \tag{3.4}$$

where

$$\boldsymbol{s}^{\boldsymbol{k}} = (3s_1 - 3b_{k_1}, \ldots, 3s_p - 3b_{k_p}), \quad k_i \in \{0, 1, \ldots, m\}, \quad \text{and} \quad b_j = \frac{2}{3}j.$$

To calculate $\tau(2)$, we observe that if ν satisfies the self-similar identity

$$\nu(E) = \sum_{j=0}^{m} c_j \nu \circ S_j^{-1}(E), \tag{3.5}$$

where $S_j(x) = \frac{1}{3}x + \frac{2}{3}j$ for $0 \le j \le m$, then according to the algorithm in the previous section, the state space is inductively defined by

$$t_n = 3t_{n-1} - 2(j_1 - j_2), \quad 0 \le j_1, j_2 \le m,$$

starting from $t_0 = 0$. Since $C = \dfrac{b_{\max}}{1-\rho} = \dfrac{2m/3}{1-1/3} = m$, we see that $t_n \in \mathcal{S}_1$ if and only if $|t_n| \le m$. By identifying the positive and negative elements (Remark 1) and omitting the boundary element according to Remark 2, we conclude that $\mathcal{S}^\sigma = \{0, 2, \ldots, 2\bar{m}\}$, where $\bar{m} = \left[\frac{m-1}{2}\right]$. Consequently for the state $2j \in \mathcal{S}^\sigma$,

$$T^\sigma(2j) = \sum_{i \in \mathcal{S}^\sigma}{}' \alpha_{ij} \cdot (2i),$$

where

$$\alpha_{ij} = \sum \big\{c_{j_1} c_{j_2} : i = |3j - (j_1 - j_2)|\big\}.$$

Let $a_k = \sum_\ell c_\ell c_{\ell-k}$. Then $a_k = a_{-k}$ and it follows that for $0 \le i, j \le \bar{m}$,

$$\alpha_{ij} = \begin{cases} a_{3j} & \text{if} \quad i = 0, \\ a_{3j+i} + a_{3j-i} & \text{if} \quad i \ne 0. \end{cases}$$

Hence the matrix T^σ is

$$
\begin{bmatrix}
a_0 & a_3 & a_6 & \cdots & a_{3\bar{m}} \\
2a_1 & a_4 + a_2 & a_7 + a_5 & \cdots & a_{3\bar{m}+1} + a_{3\bar{m}-1} \\
2a_2 & a_5 + a_1 & a_8 + a_4 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2a_{\bar{m}} & a_{\bar{m}+3} + a_{\bar{m}-3} & \cdots & \cdots & a_{4\bar{m}} + a_{2\bar{m}}
\end{bmatrix}. \tag{3.6}
$$

By using Theorem 2.4, we have

THEOREM 3.2. *Let μ be the standard Cantor measure and let $\nu = \mu^{*m}$. Then for $p = 2$, the corresponding matrix T^σ for ν is given by (3.6) with $\{a_k\} = \{c_k\} * \{c_{-k}\}$, and $c_k = \frac{1}{2^m}\binom{m}{k}$. The L^2-scaling exponent $\tau(2) = |\ln \lambda_m / \ln 3|$, where λ_m is the maximal eigenvalue of T^σ.*

The following is a list of values of $\tau(2)$ for $m \leq 10$.

m	$\tau(2)$	m	$\tau(2)$
1	0.6309297535	6	0.9997949242
2	0.8927892607	7	0.9999564485
3	0.9766281250	8	0.9999906718
4	0.9952461964	9	0.9999979911
5	0.9990215851	10	0.9999995658

In regard to $\tau(p)$ for the other integers p, we only consider the case $\nu = \mu * \mu * \mu$. We need to define three $p \times p$ matrices:

$$
A^{(0)} = \begin{bmatrix}
\binom{p}{0}(1 + 3^p) & 0 & \dots & 0 \\
\binom{p}{1}(3 + 3^p) & 0 & \dots & 0 \\
\binom{p}{2}(3^2 + 3^p) & 0 & \dots & 0 \\
\vdots & & & \vdots \\
\binom{p}{p-2}(3^{p-2} + 3^p) & 0 & \dots & 0 \\
\binom{p}{p-1}(3^{p-1} + 3^p) & 0 & \dots & 0
\end{bmatrix},
$$

$$
A^{(1)} = \begin{bmatrix}
3^p & 0 & 0 & \dots & 0 \\
3^{p-1}\binom{p}{1} & 3^{p-1}\binom{p-1}{0} & 0 & \dots & 0 \\
3^{p-2}\binom{p}{2} & 3^{p-2}\binom{p-1}{1} & 3^{p-2}\binom{p-2}{0} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
3\binom{p}{p-1} & 3\binom{p-1}{p-2} & 3\binom{p-2}{p-3} & \cdots & 3\binom{1}{0}
\end{bmatrix},
$$

$$
A^{(2)} = \begin{bmatrix}
1 & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{p-2}{0} & \binom{p-1}{0} \\
0 & 3\binom{1}{1} & 3\binom{2}{1} & \cdots & 3\binom{p-2}{1} & 3\binom{p-1}{1} \\
\vdots & 0 & 3^2\binom{2}{2} & \cdots & 3^2\binom{p-2}{2} & 3^2\binom{p-1}{2} \\
0 & 0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 3^{p-2}\binom{p-2}{p-2} & 3^{p-2}\binom{p-1}{p-2} \\
0 & 0 & 0 & \cdots & 0 & 3^{p-1}\binom{p-1}{p-1}
\end{bmatrix}.
$$

THEOREM 3.3. *Let μ be the usual Cantor measure and let $\nu = \mu * \mu * \mu$. Then for any integer $p \geq 2$, $\tau(p) = |\ln \lambda_p / \ln 3|$, where λ_p is the maximal eigenvalue of the $p \times p$ matrix*

$$A_p = \frac{1}{2^{3p}}(A^{(0)} + A^{(1)} + A^{(2)}).$$

Proof. We first apply the algorithm in Section 2 to find the set $\mathcal{S}^\sigma$. For $\boldsymbol{s}_0 = (0, \ldots, 0)$, by using (3.4) we have

$$T(\boldsymbol{s}_0) = \frac{1}{2^{3p}} \sum_{\boldsymbol{k}}{}' \binom{3}{k_1} \cdots \binom{3}{k_p} \cdot (-3b_{k_1}, \ldots, -3b_{k_p}), \tag{3.7}$$

where $b_{k_i} \in \{0, \frac{2}{3}, \frac{4}{3}, 2\}$. By observing that for $1 \leq i, j \leq p$, $3b_{k_i} - 3b_{k_j}$ are even integers, and by applying the condition $|3b_{k_i} - 3b_{k_j}| < 3$ to be a member in $\mathcal{S}^\sigma$, we conclude that the states in $\mathcal{S}^\sigma$ generated by $T(\boldsymbol{s}_0)$ are of the form

$$\boldsymbol{s}_n := (\underbrace{2, \ldots, 2}_{n}, 0, \ldots, 0), \quad 0 \leq n \leq p-1. \tag{3.8}$$

For such $\boldsymbol{s}_n$ with $1 \leq n \leq p-1$,

$$T(\boldsymbol{s}_n) = \frac{1}{2^{3p}} \sum_{\boldsymbol{k}}{}' \binom{3}{k_1} \cdots \binom{3}{k_p} \cdot (6 - 3b_{k_1}, \ldots, 6 - 3b_{k_n}, -3b_{k_{n+1}}, \ldots, -3b_{k_p}). \tag{3.9}$$

By the same argument as above, the states in $\mathcal{S}^\sigma$ generated by $T(\boldsymbol{s}_n)$ must belong to the set $\{\boldsymbol{s}_n\}_{n=0}^{p-1}$. Since no more new states are generated, we conclude that $\mathcal{S}^\sigma$ consists of the p states in (3.8).

Next, we write (3.7) and (3.9) as $T(\boldsymbol{s}_n) = \dfrac{1}{2^{3p}} \sum_\ell' \alpha_{\ell n} \cdot \boldsymbol{s}_\ell$. We want to calculate the value of each entry $\alpha_{\ell n}$ of T^σ. For the first column (corresponding to $T(\boldsymbol{s}_0)$ given by (3.7)), we can see from (3.7) that the coefficient of $\boldsymbol{s}_\ell$ $(1 \leq \ell \leq p-1)$ comes from rearranging the following three types of states:

(i) Exactly ℓ of the b_{k_i} are 0 and the rest are 2/3 (which means $k_i = 0$ and 1 respectively). The sum of the probability weights from these states equals

$$\frac{1}{2^{3p}} \binom{p}{\ell} \binom{3}{0}^\ell \binom{3}{1}^{p-\ell} = \frac{3^{p-\ell}}{2^{3p}} \binom{p}{\ell},$$

i.e. $\alpha_{\ell n} = 3^{p-\ell}\binom{p}{\ell}$, which is the corresponding entry in $A^{(1)}$.

(ii) Exactly ℓ of the b_{k_i} are 2/3 and the rest are 4/3. The probability weights from these states sum up to

$$\frac{1}{2^{3p}} \binom{p}{\ell} \binom{3}{1}^\ell \binom{3}{2}^{p-\ell} = \frac{3^p}{2^{3p}} \binom{p}{\ell}.$$

(iii) Exactly ℓ of the b_{k_i} are 4/3 and the rest are 2, with sum of probability weights equal to

$$\frac{1}{2^{3p}} \binom{p}{\ell} \binom{3}{2}^\ell \binom{3}{3}^{p-\ell} = \frac{3^\ell}{2^{3p}} \binom{p}{\ell}.$$

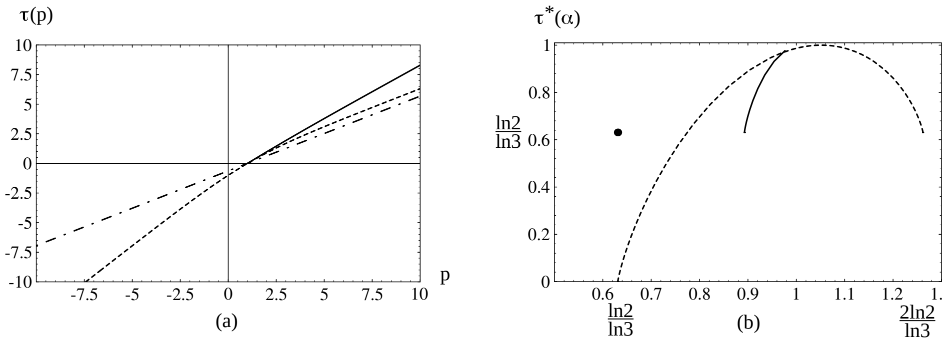

FIG. 3.1. *(a) shows curves of $\tau(p)$ for the standard Cantor measure μ together with the convolutions $\mu * \mu$ and $\mu * \mu * \mu$. For μ, $\tau(p) = \frac{\ln 2}{\ln 3}(p-1)$ (dot-dashed line). For $\mu * \mu$, $\tau(p)$ is given by (3.3) (dotted curve) and for $\mu * \mu * \mu$ it is plotted by using Theorem 3.3 (solid curve). Figure (b) shows the corresponding dimension spectra, given by $\tau^*(\alpha)$. For μ it is just the point $(\frac{\ln 2}{\ln 3}, \frac{\ln 2}{\ln 3})$. For $\mu * \mu$ it is shown by the dotted curve. In fact, it can be shown (see [LN1]) by using (3.3) that the infimum of the domain of $\tau^*(\alpha)$ is $\alpha_{\min} = \lim_{p\to\infty} \frac{\tau(p)}{p} = \frac{\ln 2}{\ln 3}$ with $\tau^*(\alpha_{\min}) = 0$, while the supremum is $\alpha_{\max} = \lim_{p\to-\infty} \frac{\tau(p)}{p} = \frac{2\ln 2}{\ln 3}$ with $\tau^*(\alpha_{\max}) = \frac{\ln 2}{\ln 3}$. The dimension spectrum for $\mu * \mu * \mu$ (solid curve) is approximated by using integral values of $\tau(p)$ for $0 \le p \le 300$. We are not able to calculate $\tau(p)$ for $p < 0$ and hence the corresponding part of $\tau^*(\alpha)$ is not shown.*

Types (ii) and (iii) together account for the first column of $A^{(0)}$. To get $\alpha_{\ell 0}$, one only has to add the case when all b_{k_i} $(1 \le i \le p)$ are 0.

To find the entries $\alpha_{\ell n}$ corresponding to $T(\boldsymbol{s}_n)$, $n \ge 1$, we use (3.9) and divide the $\alpha_{\ell n}$ into three classes:

(a) $\ell > n$. In order for the rearrangement of $(6-3b_{k_1}, \dots, 6-3b_{k_n}, -3b_{k_n+1}, \dots, -3b_{k_p})$ to equal $\boldsymbol{s}_\ell$, the following conditions must be satisfied: $b_{k_i} = 2$ for all $1 \le i \le n$, and for $n+1 \le j \le p$, exactly $\ell - n$ of the b_{k_j} are 0 and the rest are $2/3$. Hence the coefficient of $\boldsymbol{s}_\ell$ is

$$\frac{1}{2^{3p}}\binom{p-n}{\ell-n}\binom{3}{3}^n\binom{3}{0}^{\ell-n}\binom{3}{2}^{p-\ell} = \frac{3^{p-\ell}}{2^{3p}}\binom{p-n}{\ell-n}.$$

The corresponding entry is under the diagonal in the matrix $A^{(1)}$.

(b) $\ell < n$. In this case, the conditions become: $b_{k_j} = 0$ for $n+1 \le j \le p$, and for $1 \le i \le n$, exactly ℓ of the b_{k_i} are $4/3$ and the rest are 2. We hence get

$$\alpha_{\ell n} = \binom{n}{\ell}\binom{3}{2}^\ell\binom{3}{3}^{n-\ell}\binom{3}{0}^{p-n} = 3^\ell\binom{n}{\ell}.$$

The corresponding entry is above the diagonal in $A^{(2)}$.

(c) $\ell = n$. In this case, both (a) and (c) above can occur and we need only sum up their contributions. This accounts for the diagonals of $A^{(1)}$ and $A^{(2)}$ and the proof is complete. □

We can further apply Theorem 2.4 to consider the self-similar measure μ_ρ defined by the similitudes in (3.1), together with its convolution when ρ^{-1} is a P.V. number

[LN3]. For the case $\rho = (\sqrt{5}-1)/2$, the spectrum $\tau(p)$ of μ has been studied extensively ([L1], [L2], [LN2], [LN3].) In particular, a formula (in terms of a series) defining $\tau(p)$ is obtained and is verified to be valid for $0 \le p < \infty$ [LN2]. Recently, a formula for $\tau(p)$, $-\infty < p < 0$ has been obtained by Feng ([Fe1], [Fe2]).

In [L2], it is shown that for $\rho = (\sqrt{5}-1)/2$, $\tau(2) = 0.9923994\ldots$ and the associated $\mathcal{S}^\sigma$ and T^σ are respectively

$$\{0, \rho, \rho^2\} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

For $\nu = \mu_\rho * \mu_\rho$, the corresponding $\mathcal{S}^\sigma$ is:

$$\{0, \rho, 2\rho, 3\rho, 1+\rho, 1, 1-\rho, 2\rho-1, 3\rho-1, 4\rho-1, 2-\rho, 2-2\rho, 2-3\rho, 3-2\rho, 3-3\rho\}.$$

and the matrix T^σ is

$$\frac{1}{4^2}\begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 6 \\
0 & 4 & 0 & 0 & 0 & 6 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 1 & 0 \\
0 & 6 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

From this we get $\tau(2) = 0.9999864326\ldots$.

4. λ-Cantor measures. For $0 < \lambda < 1$ we consider the IFS

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{3}x + \frac{\lambda}{3}, \quad S_3(x) = \frac{1}{3}x + \frac{2}{3},$$

and let F_λ be the self-similar set associated with the three maps. This family of iterated function system and the invariant set F_λ have been considered by Kenyon [K] and Rao and Wen [RW]. To summarize their results, let $\lambda = \frac{a}{b} \in \mathbb{Q} \cap (0,1)$ with $(a,b) = 1$. If $a \equiv b \not\equiv 0 \pmod 3$, then the IFS satisfies the open set condition and F_λ contains interior points. On the other hand if $a \not\equiv b \pmod 3$, then the IFS does not satisfy the open set condition and $\dim_B(F_\lambda) < 1$, where $\dim_B(F_\lambda)$ denotes the box dimension of F_λ.

We let $\mu = \mu_\lambda$ be the λ-Cantor measure defined by

$$\mu = \frac{1}{3}\mu \circ S_1^{-1} + \frac{1}{3}\mu \circ S_2^{-1} + \frac{1}{3}\mu \circ S_3^{-1}. \tag{4.1}$$

Then $\operatorname{supp}(\mu) \subseteq [0,1]$. The above mentioned result implies in particular that if $a \not\equiv b \pmod 3$, then μ must be singular. It follows from Proposition 2.2 that if $\lambda \in \mathbb{Q}$, then the IFS has the WSP.

Throughout the rest of this section, we assume that $\lambda \in \mathbb{Q} \cap (0,1)$, and $a \not\equiv b \pmod 3$. We will use the method described in Section 2 to compute $\tau(p)$ for some interesting cases studied in [RW]. The case $a \equiv b \pmod 3$ will be discussed in Section 5.

For the two families of values $\lambda = 1 - \frac{1}{3^N}$ and $\lambda = \frac{2}{3^N}$ ($N \geq 1$), the Hausdorff dimension of the self-similar set F_λ has been calculated in [RW]. Hence we consider

Case I: $\lambda = 1 - \frac{1}{3^N}, N \geq 1$. Fix an integer $p \geq 2$. For $\boldsymbol{s} = (s_1, \ldots, s_p) \in \mathcal{S}$,

$$T(\boldsymbol{s}) = \frac{1}{3^p} \sum_{\epsilon_i \in \mathcal{E}_N}{}' (3s_1 - \epsilon_1, \ldots, 3s_p - \epsilon_p), \tag{4.2}$$

where $\mathcal{E}_N = \{0, 1 - \frac{1}{3^N}, 2\}$. Since $C = 1$, the rearrangement of $\boldsymbol{s} = (s_1, \ldots, s_p)$ belongs to $\mathcal{S}^\sigma$ if and only if

$$|s_i - s_j| < 1 \quad \text{for all} \quad 1 \leq i, j \leq p. \tag{4.3}$$

We will first determine the set $\mathcal{S}^\sigma$. Define

$$\boldsymbol{u}_{m,k} = (\underbrace{1 - \frac{1}{3^m}, \ldots, 1 - \frac{1}{3^m}}_{k}, 0, \ldots, 0), \quad m = 1, \ldots, N, \; k = 0, 1, \ldots, p.$$

(Note that $\boldsymbol{u}_{m,0} = \boldsymbol{u}_{m,p} = \boldsymbol{u}_0 = (0, \ldots, 0)$.)

Proposition 4.1. $\mathcal{S}^\sigma = \{\boldsymbol{u}_{m,k} : 1 \leq m \leq N, \; 0 \leq k < p\}$.

Proof. Consider the action of T on the three types of states below:

(i) $\quad T(\boldsymbol{u}_0) = \dfrac{1}{3^p} \displaystyle\sum_{\epsilon_i \in \mathcal{E}_N}{}' (\epsilon_1, \ldots, \epsilon_p).$

In the case that at least one ϵ_i from $(\epsilon_1, \ldots, \epsilon_p)$ is zero, then for the rearrangement of $(\epsilon_1, \ldots, \epsilon_p)$ to belong to $\mathcal{S}^\sigma$, condition (4.3) implies that the other ϵ_j's must be 0 or $1 - \frac{1}{3^N}$. After a rearrangement, there are $\binom{p}{k}$ states of the form $\boldsymbol{u}_{N,k}$ for each $k = 0, 1, \ldots, p-1$. On the other hand, if $\epsilon_i \neq 0$ for all $1 \leq i \leq p$, then for $(\epsilon_1, \ldots, \epsilon_p)$ to belong to $\mathcal{S}^\sigma$ after a rearrangement, either $\epsilon_i = 1 - \frac{1}{3^N}$ for all i, or $\epsilon_i = 2$ for all i. We conclude that

$$T^\sigma(\boldsymbol{u}_0) = \frac{1}{3^p} \Bigg(3 \cdot \boldsymbol{u}_0 + \sum_{k=1}^{p-1}{}' \binom{p}{k} \cdot \boldsymbol{u}_{N,k} \Bigg). \tag{4.4}$$

(ii) For $1 < m \leq N$ and $1 \leq k \leq p-1$,

$$\begin{aligned} T(\boldsymbol{u}_{m,k}) &= \frac{1}{3^p} \sum_{\boldsymbol{\epsilon}}{}' (3\boldsymbol{u}_{m,k} - \boldsymbol{\epsilon}) \\ &= \frac{1}{3^p} \sum_{\epsilon_i \in \mathcal{E}_N}{}' (3 - \frac{1}{3^{m-1}} - \epsilon_1, \ldots, 3 - \frac{1}{3^{m-1}} - \epsilon_k, -\epsilon_{k+1}, \ldots, -\epsilon_p). \end{aligned}$$

For the rearrangement of $3\boldsymbol{u}_{m,k} - \boldsymbol{\epsilon}$ to belong to $\mathcal{S}^\sigma$, condition (4.3) implies that $\epsilon_i = 2$ for $1 \leq i \leq k$, and $\epsilon_j = 0$ for $k+1 \leq j \leq p$. Consequently,

$$T^\sigma(\boldsymbol{u}_{m,k}) = \frac{1}{3^p} \cdot \boldsymbol{u}_{m-1,k}. \tag{4.5}$$

(iii) For $m = 1$ and $1 \le k \le p-1$,

$$T(\boldsymbol{u}_{1,k}) = \frac{1}{3^p} \sum_{\boldsymbol{\epsilon}}{}' (3\boldsymbol{u}_{1,k} - \boldsymbol{\epsilon})$$
$$= \frac{1}{3^p} \sum_{\epsilon_i \in \mathcal{E}_N}{}' (2-\epsilon_1, \dots, 2-\epsilon_k, -\epsilon_{k+1}, \dots, -\epsilon_p).$$

For $3\boldsymbol{u}_{1,k} - \boldsymbol{\epsilon}$ to belong to $\mathcal{S}^\sigma$ after a rearrangement, (4.3) forces $\epsilon_i = 2$ for $1 \le i \le k$, and $\epsilon_j = 0$ or $1 - \frac{1}{3^N}$ for $k+1 \le j \le p$. If ℓ $(0 \le \ell \le p-k)$ of the ϵ_j $(k+1 \le j \le p)$ are equal to $1 - \frac{1}{3^N}$, then we have $\binom{p-k}{\ell}$ states of the form $\boldsymbol{u}_{N,p-\ell}$, i.e.,

$$T^\sigma(\boldsymbol{u}_{1,k}) = \frac{1}{3^p} \sum_{\ell=0}^{p-k}{}' \binom{p-k}{\ell} \cdot \boldsymbol{u}_{N,p-\ell}. \tag{4.6}$$

This completes the proof. □

We arrange the basis elements in $\mathcal{S}^\sigma$ in the order

$$\boldsymbol{u}_0, \boldsymbol{u}_{N,p-1}, \dots, \boldsymbol{u}_{N,1}, \boldsymbol{u}_{N-1,p-1}, \dots, \boldsymbol{u}_{N-1,1}, \dots, \boldsymbol{u}_{2,p-1}, \dots, \boldsymbol{u}_{2,1}, \boldsymbol{u}_{1,p-1}, \dots, \boldsymbol{u}_{1,1}.$$

Then the proof of the above proposition also gives us the explicit form of the matrix T^σ. (4.4) and (4.6) imply that $T^\sigma(\boldsymbol{u}_0)$ and $T^\sigma(\boldsymbol{u}_{1,k})$ can be represented respectively by $\frac{1}{3^p}C_p$ and $\frac{1}{3^p}A_p$ where

$$C_p = \begin{bmatrix} 3 \\ \binom{p}{1} \\ \binom{p}{2} \\ \vdots \\ \binom{p}{p-1} \end{bmatrix}, \quad \text{and} \quad A_p = \begin{bmatrix} \binom{1}{0} & \binom{2}{0} & \cdots & \binom{p-2}{0} & \binom{p-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \cdots & \binom{p-2}{1} & \binom{p-1}{1} \\ 0 & \binom{2}{2} & \cdots & \binom{p-2}{2} & \binom{p-1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{p-2}{p-2} & \binom{p-1}{p-2} \\ 0 & 0 & \cdots & 0 & \binom{p-1}{p-1} \end{bmatrix}. \tag{4.7}$$

(Note that A_p is a $p \times (p-1)$ matrix.) (4.5) implies that $T^\sigma(\boldsymbol{u}_{m,k})$ $(1 < m \le N$ and $1 \le k \le p-1)$ can be represented as the identity matrix $I_{(N-1)(p-1)}$. For $N \ge 1$, we define an $\big(N(p-1)+1\big) \times \big(N(p-1)+1\big)$ matrix

$$M_p^{(N)} = \frac{1}{3^p} \begin{bmatrix} C_p & \mathbf{0} & A_p \\ \mathbf{0} & I_{(N-1)(p-1)} & \mathbf{0} \end{bmatrix}. \tag{4.8}$$

We have the following

THEOREM 4.2. *Let $\lambda = 1 - \frac{1}{3^N}$ $(N \ge 1)$ and let $\mu = \mu_\lambda$ be defined by (4.1). Then for any integer $p \ge 2$, the matrix T^σ for μ is equal to $M_p^{(N)}$ and $\tau(p) = |\ln \lambda_p/(p \ln 3)|$, where λ_p is the maximal eigenvalue of $M_p^{(N)}$.*

For example when $\lambda = \dfrac{2}{3}$, the matrices T^σ corresponding to $p = 2$ and $p = 3$ are respectively

$$\frac{1}{9}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \frac{1}{27}\begin{bmatrix} 3 & 1 & 1 \\ 3 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}.$$

For the corresponding μ_λ, $\tau(2) = 0.80125326\ldots$ and $\tau(3) = 1.50972657\ldots$. For $\lambda = \dfrac{8}{9}$, the matrices corresponding to $p = 2$ and $p = 3$ are respectively

$$\frac{1}{9}\begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \frac{1}{27}\begin{bmatrix} 3 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

For the corresponding μ_λ, $\tau(2) = 0.93719034\ldots$ and $\tau(3) = 1.83901647\ldots$.

Case II: $\lambda = \frac{2}{3^N}$, $N \geq 1$. As this case is more complicated in notation, we will only explain the simpler situation $p = 2$ and make a remark on the extension of this to $p \geq 2$. Define a set of states: $\boldsymbol{u}_0 := 0$, $\boldsymbol{u}_N := \dfrac{2}{3^N}$, and for $1 \leq k \leq N-1$,

$$\boldsymbol{u}_{k,\boldsymbol{\eta}} := \frac{2}{3^k} + \frac{2}{3^{k+1}}\eta_{k+1} + \cdots + \frac{2}{3^N}\eta_N,$$

where $\boldsymbol{\eta} = (\eta_{k+1}, \ldots, \eta_N)$, $\eta_i = 0, \ \pm 1, \ k+1 \leq i \leq N$.

Proposition 4.3. *Let $B_0 = \{\boldsymbol{u}_0\}$, $B_N = \{\boldsymbol{u}_N\}$, and for $1 \leq k \leq N-1$, let*

$$B_k = \Big\{\boldsymbol{u}_{k,\boldsymbol{\eta}} : \eta_i = 0, \pm 1, \ k+1 \leq i \leq N\Big\}.$$

Then for $p = 2$, $\mathcal{S}^\sigma = \bigcup_{k=0}^N B_k$. Consequently $\mathcal{S}^\sigma$ contains $(3^N+1)/2$ elements.

Proof. We will use the notation in Remark 1 of Section 2. Consider the following cases:
(i)

$$T(\boldsymbol{u}_0) = \frac{1}{9}\sum_{\epsilon_1,\epsilon_2 \in \mathcal{E}_N}{}' (-\epsilon_1 + \epsilon_2),$$

where $\mathcal{E}_N = \{0, \frac{2}{3^N}, 2\}$. If the rearrangement of $-\epsilon_1 + \epsilon_2$ belongs to $\mathcal{S}^\sigma$, then the rearranged state must be either $\boldsymbol{u}_0$ or $\boldsymbol{u}_N = \frac{2}{3^N}$, which belongs to $B_0 \bigcup B_N$.

(ii) For $2 \leq k \leq N$,

$$T(\boldsymbol{u}_{k,\boldsymbol{\eta}}) = \frac{1}{9}\sum_{\epsilon_1,\epsilon_2 \in \mathcal{E}_N}{}' \Big(\frac{2}{3^{k-1}} + \frac{2}{3^k}\eta_{k+1} + \cdots + \frac{2}{3^{N-1}}\eta_N - \epsilon_1 + \epsilon_2\Big).$$

If $\epsilon_2 = 2$, then condition (4.3) implies that ϵ_1 must also be equal to 2, and vice versa. In fact, if $\epsilon_1 = \epsilon_2$, then the corresponding state is

$$\frac{2}{3^{k-1}} + \frac{2}{3^k}\eta_{k+1} + \cdots + \frac{2}{3^{N-1}}\eta_N,$$

which belongs to B_{k-1}. The remaining choices for ϵ_1 and ϵ_2 are $(\epsilon_1, \epsilon_2) = (0, \frac{2}{3^N})$ or $(\frac{2}{3^N}, 0)$. In both cases, the rearrangement of the state is of the form

$$\frac{2}{3^{k-1}} + \frac{2}{3^k}\eta_{k+1} + \cdots + \frac{2}{3^{N-1}}\eta_N + \frac{2}{3^N}\eta_{N+1},$$

which belongs to B_{k-1}.

(iii) $k = 1$. Then

$$T(\boldsymbol{u}_{1,\boldsymbol{\eta}}) = \frac{1}{9} \sum_{\epsilon_1,\epsilon_2 \in \mathcal{E}_N}{}' \left(2 + \frac{2}{3}\eta_2 + \cdots + \frac{2}{3^{N-1}}\eta_N - \epsilon_1 + \epsilon_2\right).$$

For the rearrangement of $2 + \frac{2}{3}\eta_2 + \cdots + \frac{2}{3^{N-1}}\eta_N - \epsilon_1 + \epsilon_2$ to belong to $\mathcal{S}^\sigma$, condition (4.3) forces $\epsilon_1 = 2$, and $\epsilon_2 = 0$ or $\frac{2}{3^N}$. The corresponding state is

$$\frac{2}{3}\eta_2 + \cdots + \frac{2}{3^{N-1}}\eta_N + \frac{2}{3^N}\eta_{N+1}, \quad \eta_{N+1} = 0 \text{ or } 1.$$

The rearrangement of this state clearly belongs to $\displaystyle\bigcup_{k=0}^{N} B_k$.

The assertion $\mathcal{S}^\sigma = \bigcup_{k=0}^{N} B_k$ follows by combining (i), (ii) and (iii). □

By using the above we can easily write down the respective matrices T^σ for $\lambda = \dfrac{2}{3}$ and $\dfrac{2}{3^2}$:

$$\frac{1}{9}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \frac{1}{9}\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

For the corresponding μ_λ, $\tau(2) = 0.80125326\ldots$ and $0.86881773\ldots$ respectively.

Remark. The above proof can be generalized to the case $p \geq 2$. For $N \geq 1$ and for $1 \leq m \leq N-1$, define

$$\boldsymbol{u}_{m,\boldsymbol{\epsilon}} = \frac{2}{3^m} + \frac{2}{3^{m+1}}\epsilon_{m+1} + \cdots + \frac{2}{3^N}\epsilon_N,$$

where $\boldsymbol{\epsilon} = (\epsilon_{m+1}, \ldots, \epsilon_N),\ \epsilon_i \in \mathcal{E}_N = \{0, \frac{2}{3^N}, 2\}$. Also, we let

$$\boldsymbol{u}_{N,\boldsymbol{\epsilon}} = \frac{2}{3^N} \quad \text{and} \quad \boldsymbol{u}_{N+1,\boldsymbol{\epsilon}} = 0, \quad \text{for all } \boldsymbol{\epsilon}.$$

It can be shown that $\mathcal{S}^\sigma$ is the set consisting of all states of the form

$$(\boldsymbol{u}_{m_1,\boldsymbol{\epsilon}_1}, \ldots, \boldsymbol{u}_{m_k,\boldsymbol{\epsilon}_k}, 0 \ldots, 0),$$

where $\boldsymbol{u}_{m_1,\boldsymbol{\epsilon}_1} \geq \cdots \geq \boldsymbol{u}_{m_k,\boldsymbol{\epsilon}_k} > 0$, $\boldsymbol{\epsilon}_i \in \mathcal{E}_N^{N-m_i}$, $1 \leq m_i \leq N+1$, and $0 \leq k \leq p-1$. We omit the proof since it is similar to the one above.

5. Fourier transformation. Let μ be a bounded Borel measure on $\mathbb{R}$ and let $\hat{\mu}(\xi) = \displaystyle\int_{-\infty}^{\infty} e^{2\pi it\xi}\, d\mu(t)$ be the Fourier transform of μ. The following general Tauberian theorem shows that the L^2-scaling exponent $\tau(2)$ also plays an important role in Fourier transformation [LW, Corollary 4.5].

Theorem 5.1. *Let μ be a bounded positive Borel measure on $\mathbb{R}$ with compact support and let*

$$\Phi^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} |\mu(B_h(x))|^2 \, dx$$

as defined in Section 2. Suppose $0 < \overline{\lim}_{h\to 0^+} \Phi^{(\alpha)}(h) < \infty$. *Then there exist* $C_1, C_2 > 0$ *such that*

$$C_1 \overline{\lim_{h\to 0^+}} \Phi^{(\alpha)}(h) \le \overline{\lim_{T\to\infty}} \frac{1}{T^{1-\alpha}} \int_{|\xi|<T} \left|\hat{\mu}(\xi)\right|^2 d\xi \le C_2 \overline{\lim_{h\to 0^+}} \Phi^{(\alpha)}(h).$$

For a self-similar measure defined by an equicontractive iterated function system such that the set of states $\mathcal{S}_1$ is finite, if we denote $\tau(2)$ by α, then $\Phi^{(\alpha)}(h)$ is an asymptotically multiplicative periodic function with period ρ (see (2.8)). This implies $0 < \overline{\lim}_{h\to 0^+} \Phi^{(\alpha)}(h) < \infty$ and the rate of $\int_{|\xi|\le T} |\hat{\mu}(\xi)|^2 d\xi$ is $T^{1-\tau(2)}$ as $T \to \infty$.

If $\tau(2) = 1$, then $0 < \overline{\lim}_{h\to 0^+} \Phi^{(\tau(2))}(h) = \overline{\lim}_{h\to 0^+} \frac{1}{h^2} \int_{-\infty}^{\infty} |\mu(B_h(x))|^2 \, dx < \infty$ implies that μ is absolutely continuous with an L^2-derivative ([HL]). It follows that $\int_{|\xi|<T} |\hat{\mu}(\xi)|^2 d\xi$ tends to the constant $\|\hat{\mu}\|_2^2$ as $T \to \infty$. This does not provide sufficient information regarding the rate that $\hat{\mu}$ converges to zero at ∞. A more effective way is to consider $\int_{|\xi|<T} |\xi|^s |\hat{\mu}(\xi)|^2 d\xi$ for suitable s. More generally in the following we will consider the rate of $\int_{|\xi|<T} |\xi|^s |\hat{\mu}(\xi)|^q d\xi$ for $q > 0$.

For a self-similar measure μ defined by (2.1) with contraction ratio ρ, its Fourier transform is

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} P(\rho^n \xi),$$

where $P(\xi) = \sum_{j=0}^{m} w_j e^{2\pi i b_j \xi/\rho}$. When ρ^{-1} is an integer > 1, there is an elegant theory due to Ruelle for handling this infinite product and the integral of $\hat{\mu}$ ([Bo], [FL]). Let g be a nonnegative Hölder continuous 1-periodic function such that $g(0) = 1$. In order to study the class of λ-Cantor measures, we define a Ruelle operator L_g on the space of continuous functions $C[0,1]$ as

$$L_g f(x) = g(\frac{x}{3}) f(\frac{x}{3}) + g(\frac{x}{3} + \frac{1}{3}) f(\frac{x}{3} + \frac{1}{3}) + g(\frac{x}{3} + \frac{2}{3}) f(\frac{x}{3} + \frac{2}{3}).$$

The spectral radius of L_g, which is the maximal eigenvalue of the positive operator L_g, is given by $\rho_g = \lim_{n\to\infty} \|L_g^n 1\|^{1/n}$ (supremum norm). The reader can refer to [Bo] for the significance of ρ_g in connection with the dynamical system defined by L_g. For $q > 0$, we let $\rho(q) = \rho_{g^q}$ where $g^q(x) := \big(g(x)\big)^q$.

Theorem 5.2. *Let g be a strictly positive, Hölder continuous 1-periodic function on $\mathbb{R}$ with $g(0) = 1$ and let $\rho(q)$ be the spectral radius of ρ_{g^q}. Then for* $G(\xi) =$

$\prod_{k=1}^{\infty} g(\frac{\xi}{3^k})$ *and* $\alpha = s + \dfrac{\ln \rho(q)}{\ln 3}$,

$$\int_1^T \xi^s G(\xi)^q d\xi \approx \begin{cases} T^\alpha & \text{if } \alpha > 0 \\ \ln T & \text{if } \alpha = 0 \\ O(1) & \text{if } \alpha < 0. \end{cases}$$

If $G_1(\xi) = \prod_{k=1}^{\infty} g_1(\frac{\xi}{3^k})$ *where* $g_1(\xi) = \big(\frac{1}{3}(1 + e^{2\pi i\xi} + e^{4\pi i\xi})\big)^N g(\xi)$ *(*g_1 *admits zeroes at* $\xi = \dfrac{1}{3}$ *and* $\dfrac{2}{3}$*), then for* $\alpha = s + \dfrac{\ln \rho(q)}{\ln 3} - Nq$*, the integral* $\displaystyle\int_1^T \xi^s |G_1(\xi)|^q d\xi$ *has the same expression as above.*

(The sign $\approx$ means the left and the right hand sides dominate each other by positive constants.)

Proof. The first part is proved in [FL, Theorem 3.2] for $\rho = \frac{1}{2}$. The proof for $\rho = \frac{1}{3}$ here is the same. For the second part we observe that

$$h(\xi) := \frac{1}{3}(1 + e^{2\pi i\xi} + e^{4\pi i\xi}) = \frac{1}{3} \cdot \frac{1 - e^{6\pi i\xi}}{1 - e^{2\pi i\xi}}.$$

Hence

$$\prod_{k=1}^{n} h(\frac{\xi}{3^k}) = \frac{1}{3^n} \cdot \frac{1 - e^{2\pi i\xi}}{1 - e^{2\pi i\xi/3^n}},$$

so that $\displaystyle\prod_{k=1}^{\infty} \Big| h(\frac{\xi}{3^k}) \Big| = \Big| \frac{1 - e^{2\pi i\xi}}{2\pi i\xi} \Big| = \frac{|\sin \pi\xi|}{\pi|\xi|}$. It follows that

$$\prod_{k=1}^{\infty} \Big| g_1(\frac{\xi}{3^k}) \Big| = \Big(\frac{|\sin \pi\xi|}{\pi|\xi|} \Big)^N \prod_{k=1}^{\infty} g(\frac{\xi}{3^k}).$$

We can apply the same argument as in [FL, Theorem 3.4] and conclude the second part of the assertion. □

The value $s := s_0$ for which $\alpha = 0$ is significant. We let

$$\beta(q) = \sup\Big\{ t : \int_{\mathbb{R}} (1 + |\xi|^q)^t G(\xi)^q d\xi < \infty \Big\}. \tag{5.1}$$

Note that $\beta(2)$ is the Sobolev exponent of the function (or distribution) F satisfying $\hat{F} = G$. It is easy to show that for the G in Theorem 5.2, $\beta(q) = s_0 = -\dfrac{\ln \rho(q)}{q \ln 3}$, and for G_1, $\beta(q) = s_0 = -\dfrac{\ln \rho(q)}{q \ln 3} + N$. Moreover, it follows from the comments following Theorem 5.1 that for $\tau(2) < 1$, $\beta(2) = (\tau(2) - 1)/2$.

We will now make use of this to consider the class of λ-Cantor measures $\mu = \mu_\lambda$ in (4.1), where $\lambda = \dfrac{a}{b}$, $0 < a < b$ are integers and $(a, b) = 1$. It follows that

$$\hat{\mu}(\xi) = \prod_{k=1}^{\infty} P(\frac{\xi}{3^k})$$

with $P(\xi) = \frac{1}{3}(1 + e^{2\pi i\lambda\xi} + e^{4\pi i\xi})$. Note that $P(b\xi) = \frac{1}{3}(1 + e^{2\pi i a\xi} + e^{4\pi i b\xi})$. It is more convenient to replace $P(b\xi)$ by

$$Q(z) = \begin{cases} 1 + z^a + z^{2b}, & \text{if } a \text{ is odd,} \\ 1 + z^{\frac{a}{2}} + z^b, & \text{if } a \text{ is even.} \end{cases} \tag{5.2}$$

We need the following technical proposition. The proof of it is completely algebraic and will be postponed to the end of the section.

Proposition 5.3. *Let Q be defined as above. We have*
(i) if $a \not\equiv b$ (mod 3), then Q has no root with $|z| = 1$;
(ii) if $a \equiv b$ (mod 3), then Q has only two simple roots with $|z| = 1$, namely $e^{2\pi i/3}$ and $e^{4\pi i/3}$.

By using this proposition we can immediately conclude from Theorem 5.2 that

Theorem 5.4. *Let μ be the self-similar measure defined by (4.1) and let $\beta(q)$ be the exponent in (5.1) for $G := \hat{\mu}$. Then the following hold:*
(i) if $a \not\equiv b$ (mod 3), then

$$\beta(q) = -\ln \rho(q)/q \ln 3,$$

where $\rho(q)$ is the spectral radius of L_{g^q} with $g(\xi) = \dfrac{1}{3}|Q(e^{2\pi i\xi})|$;
(ii) if $a \equiv b$ (mod 3), then

$$\beta(q) = 1 - \ln \rho(q)/q \ln 3,$$

where $\rho(q)$ is the spectral radius corresponding to g^q with

$$g(\xi) = |Q(e^{2\pi i\xi})|/(1 + e^{2\pi i\xi} + e^{4\pi i\xi}).$$

In either case $\displaystyle\int_{|\xi|<T} (1 + |\xi|^q)^s |\hat{\mu}(\xi)|^q \, d\xi$ has the expression as in Theorem 5.2 with $\alpha = q(s - \beta(q))$.

Proof. Assume $a \not\equiv b$ (mod 3). If a is odd, then $|\hat{\mu}(b\xi)| = \prod_{k=1}^{\infty} g(\frac{\xi}{3^k})$, where $g(\xi) = \dfrac{1}{3}|Q(e^{2\pi i\xi})|$, and if a is even, then $\hat{\mu}(2b\xi) = \prod_{k=1}^{\infty} g(\frac{\xi}{3^k})$. Hence we need only consider $\prod_{k=1}^{\infty} g(\frac{\xi}{3^k})$. Since g is strictly positive, the first part of Theorem 5.2 applies. If $a \equiv b$ (mod 3), then g in the alternative form is strictly positive and the second part of Theorem 5.2 applies. □

In the following we want to calculate the spectral radius of L_g and the exponents in Theorem 5.4. We make use of the following observation of Hervé [H]. Suppose F is an invariant subspace of L_g in $C[0,1]$ and contains the constant function 1. Then L_g and $L_g|_F$ have the same spectral radius. The most interesting case is when g is a positive trigonometric polynomial. We will see that we can take F to be a finite dimensional subspace of trigonometric polynomials.

Let $g(x) = \sum_{n=-N}^{N} a_n e^{2\pi i n x}$. We decompose g in the following manner

$$\begin{aligned} g(x) &= e^{-2\pi i x} \sum a_{3n-1} e^{6\pi i n x} + \sum a_{3n} e^{6\pi i n x} + e^{2\pi i x} \sum a_{3n+1} e^{6\pi i n x} \\ &:= e^{-2\pi i x} g_{-1}(3x) + g_0(3x) + e^{2\pi i x} g_1(3x). \end{aligned}$$

Similarly for any trigonometric polynomial f, we can decompose

$$f(x) = e^{-2\pi ix} f_{-1}(3x) + f_0(3x) + e^{2\pi ix} f_1(3x).$$

It follows that

$$\begin{aligned} L_g f(x) &= \sum_k f(\frac{x}{3}+\frac{k}{3}) g(\frac{x}{3}+\frac{k}{3}) \\ &= \sum_k \sum_{j_1,j_2} e^{2\pi i(j_1+j_2)(\frac{x}{3}+\frac{k}{3})} f_{j_1}(x) g_{j_2}(x) \\ &= \sum_{j_1,j_2} \Big(e^{2\pi i(j_1+j_2)\frac{x}{3}} f_{j_1}(x) g_{j_2}(x) \sum_k e^{2\pi ik(j_1+j_2)/3} \Big), \end{aligned}$$

where the sums of k, j_1, j_2 are over $-1, 0, 1$. Note that the last term is

$$\sum_k e^{2\pi ik(j_1+j_2)/3} = \begin{cases} 0 & \text{if } \frac{j_1+j_2}{3} \neq 0, \\ 3 & \text{if } \frac{j_1+j_2}{3} = 0. \end{cases}$$

We conclude that

$$L_g f(x) = 3\Big(g_0(x) f_0(x) + g_1(x) f_{-1}(x) + g_{-1}(x) f_1(x) \Big). \tag{5.3}$$

From this we see that L_g is invariant on the space of trigonometric polynomials $\boldsymbol{T}_{\bar{N}}$ of degree not greater than $\bar{N}$ where $\bar{N} = [N/2] + 1$ for $N = 2$ and $\bar{N} = [(N-1)/2]$ for $N \geq 3$.

By using (5.3), it is easy to see that if $f(x) = e^{2\pi i(3\ell x)}$, then

$$L_g f(x) = 3 g_0(x) e^{2\pi i\ell x} = 3 \sum_n a_{3n-3\ell} e^{2\pi inx};$$

if $f(x) = e^{2\pi i(3\ell-1)x}$, then

$$L_g f(x) = 3 g_1(x) e^{2\pi i\ell x} = 3 \sum_n a_{3n-3\ell+1} e^{2\pi inx};$$

if $f(x) = e^{2\pi i(3\ell+1)x}$, then

$$L_g f(x) = 3 g_{-1}(x) e^{2\pi i\ell x} = 3 \sum_n a_{3n-3\ell-1} e^{2\pi inx}.$$

Consequently for the basis $\{e^{2\pi inx}\}_{|n|\leq \bar{N}}$, we can write down the $(2\bar{N}+1)\times(2\bar{N}+1)$ matrix representing L_g:

$$3 \begin{bmatrix} & & & \vdots & & & \\ & a_{-4} & a_{-5} & a_{-6} & a_{-7} & a_{-8} & \\ & a_{-1} & a_{-2} & a_{-3} & a_{-4} & a_{-5} & \\ \cdots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ & a_5 & a_4 & a_3 & a_2 & a_1 & \\ & a_8 & a_7 & a_6 & a_5 & a_4 & \\ & & & \vdots & & & \end{bmatrix}.$$

If $g(x) = \sum_{n=-N}^{N} a_n e^{2\pi inx}$ is such that $a_n = a_{-n}$, then $g(x) = a_0 + 2\sum_{n=1}^{N} a_n \cos(2\pi nx)$. If we use $\cos 2\pi nx = \frac{1}{2}(e^{2\pi inx} + e^{-2\pi inx})$, $0 \le n \le \bar{N}$ as a basis, we can obtain, directly from above, an $(\bar{N}+1) \times (\bar{N}+1)$ matrix representing L_g,

$$M = 3\begin{bmatrix} a_0 & 2a_1 & 2a_2 & \cdots & 2a_{\bar{N}} \\ a_3 & a_4 + a_2 & a_5 + a_1 & \cdots & a_{\bar{N}+3} + a_{\bar{N}-3} \\ a_6 & a_7 + a_5 & a_8 + a_4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{3\bar{N}} & a_{3\bar{N}+1} + a_{3\bar{N}-1} & \cdots & \cdots & a_{4\bar{N}} + a_{2\bar{N}} \end{bmatrix}.$$

We remark that for μ given by (3.5), $\hat{\mu}(\xi) = \prod_{n=1}^{\infty} P(\frac{\xi}{3^n})$ where $P(\xi) = \sum_{k=0}^{m} c_k e^{4\pi ik\xi}$. Hence

$$\begin{aligned} g(\xi) :=& \big|P(\xi)\big|^2 = \Big(\sum_k c_k e^{4\pi ik\xi}\Big)\Big(\sum_j c_j e^{-4\pi ij\xi}\Big) \\ &= \sum a_n e^{4\pi in\xi}, \end{aligned}$$

where $a_n = \sum_\ell c_\ell c_{\ell-n}$ and $a_n = a_{-n}$. The matrix corresponding to $g(\xi/2) = \sum a_n e^{2\pi in\xi}$ is the transpose of that in (3.6). This is necessary in view of Theorem 5.1 and Theorem 5.2 with $s = 0$.

Return to the case in (5.2). We see from the proof of Theorem 5.4 that we need only consider $|g(\xi)|^2$, where $g(\xi) = \frac{1}{3}|Q(e^{2\pi i\xi})|$ with Q defined by (5.2).

Case 1. $a \not\equiv b \pmod 3$. In this case μ is singular, we have calculations for $\rho(2)$, $\beta(2)$ (Sobolev exponent), and $\tau(2)$ (L^2-scaling exponent) for some simple cases.
(i) $\lambda = \frac{1}{2}$: $|g(\xi)|^2 = \frac{1}{3} + \frac{2}{9}\cos 2\pi\xi + \frac{2}{9}\cos 6\pi\xi + \frac{2}{9}\cos 8\pi\xi$. The M corresponding to $L_{|g|^2}$ is

$$\begin{bmatrix} 1 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \rho(2) &= \rho_{|g|^2} = \frac{2+\sqrt{3}}{3} = 1.24402, \\ \beta(2) &= -\frac{\ln \rho(2)}{2\ln 3} = -0.099373 \qquad \text{(Theorem 5.4)}, \\ \tau(2) &= 1 - \frac{\ln \rho(2)}{\ln 3} = 1 + 2\beta(2) = 0.801253 \qquad \text{(Theorem 5.1)}. \end{aligned}$$

(Note that $\tau(2)$ can also be obtained by Theorem 2.3.)
(ii) $\lambda = \frac{1}{3}$: $|g(\xi)|^2 = \frac{1}{3} + \frac{2}{9}\cos 2\pi\xi + \frac{2}{9}\cos 10\pi\xi + \frac{2}{9}\cos 12\pi\xi$.

$$M = \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 2/3 \\ 1/3 & 1/3 & 0 \end{bmatrix}.$$

$$\rho(2) = \frac{2+\sqrt{2}}{3} = 1.13807, \ \beta(2) = -0.058863, \ \tau(2) = 0.882274.$$

(iii) $\lambda = \frac{2}{3}$: $|g(\xi)|^2 = \frac{1}{3} + \frac{2}{9}\cos 2\pi\xi + \frac{2}{9}\cos 4\pi\xi + \frac{2}{9}\cos 6\pi\xi$.
Since the matrix is the same as case (i), the values are the same.

Case 2. $a \equiv b \pmod 3$. In this case μ is absolutely continuous and hence $\tau(2) = 1$. In view of Theorem 5.4, we replace $|g(\xi)|^2$ by

$$|g(\xi)|^2 = \frac{|Q(e^{2\pi i\xi})|^2}{|1 + e^{2\pi i\xi} + e^{4\pi i\xi}|^2}.$$

(i) $\lambda = \frac{1}{4}$: By using long division, we get

$$\frac{Q(z)}{1+z+z^2} = \frac{1+z+z^8}{1+z+z^2} = 1 - z^2 + z^3 - z^5 + z^6.$$

Hence

$$|g(\xi)|^2 = 5 - 4\cos 2\pi\xi - 4\cos 4\pi\xi + 6\cos 6\pi\xi - 2\cos 8\pi\xi - 2\cos 10\pi\xi + 2\cos 12\pi\xi.$$

$$M = \begin{bmatrix} 15 & -12 & -12 \\ 9 & -9 & -9 \\ 3 & -3 & -3 \end{bmatrix}, \qquad \rho_1(2) = 7.68466, \qquad \beta(2) = 0.071908.$$

(ii) $\lambda = \frac{2}{5}$: $|g(\xi)|^2 = 3 - 2\cos 2\pi\xi - 2\cos 4\pi\xi + 2\cos 6\pi\xi$.

$$M = \begin{bmatrix} 9 & -6 \\ 3 & -3 \end{bmatrix}, \qquad \rho_1(2) = 7.24264, \qquad \beta(2) = 0.0988696.$$

The following is a table of the Sobolev exponents $\beta(2)$ of μ with $\lambda = \frac{a}{b}$, a rational number. When the exponent is negative, the corresponding measure is singular; otherwise it is absolutely continuous. (Note that for $\tau(2) < 1$, $\beta(2) = (\tau(2) - 1)/2$.)

	a=1	a=2	a=3	a=4	a=5	a=6	a=7	a=8	a=9	a=10
b= 2	-.0994									
3	-.0589	-.0994								
4	.0719	/	-.0466							
5	-.0994	.0989	-.0600	-.0440						
6	-.0444	/	/	/	-.0273					
7	.0405	-.0734	-.0355	.0601	-.0485	-.0288				
8	-.0410	/	-.0328	/	0.0200	/	-.0239			
9	-.0598	-.0994	/	-.0734	-.0469	/	-.0305	-.0314		
10	.0301	/	-.0422	/	/	/	.0114	/	-0.197	
11	-.0378	.0558	-.0345	-.0561	.0155	-.0490	-.0362	.0236	-.0262	-.0186

Finally we will complete the proof of Proposition 5.3. We need a lemma.

LEMMA 5.5. *Let z and γ be two complex numbers with $|z| = |\gamma| = 1$ and satisfy $1 + z + \gamma z^2 = 0$. Then $\gamma = 1$, and $z = e^{2\pi i/3}$ or $e^{4\pi i/3}$.*

Proof. The hypothesis implies that $1 = |1 + \gamma z|$, which reduces to $2\mathrm{Re}(\gamma z) = -1$. Hence $\gamma z = \alpha$ or α^2 where $\alpha = e^{2\pi i/3}$. Substituting $z = \alpha/\gamma$ or α^2/γ into $1 + z + \gamma z^2 = 0$ yields $\gamma = 1$ and the lemma follows. □

Proof of Proposition 5.3. We will first consider the case when a is odd. Let $|z|=1$ be a root of Q. Then

$$1+z^a+z^{2(b-a)}z^{2a}=0.$$

Lemma 5.5 implies that $z^{2(b-a)}=1$ and

$$(1)\;\; z^a=e^{2\pi i/3}:=\alpha \quad \text{or} \quad (2)\; z^a=e^{4\pi i/3}=\alpha^2. \tag{5.4}$$

We concentrate on case (1); case (2) follows by using $\bar{z}$ instead. By writing $z=e^{2\pi ix}$, we get $\frac{1}{a}(\frac{1}{3}+k_1)=x=\frac{k_2}{2(b-a)}$ for some integers k_1 and k_2. It follows that

$$2(b-a)(1+3k_1)=3ak_2 \tag{5.5}$$

and hence $a\equiv b \pmod 3$. This proves part (i) of the proposition. Also it is a direct check that both α and α^2 are roots of Q if $a\equiv b \pmod 3$. To see that α and α^2 are simple roots, it suffices to note that $Q'(\alpha)\neq 0$ and $Q'(\alpha^2)\neq 0$.

We next show that in the case $a\equiv b \pmod 3$, α and α^2 are the only roots of Q of modulus 1. Observe from (5.5) that k_2 must be even (since a is odd). Hence $2(b-a)x=k_2$ implies that $(b-a)x$ is an integer and $z^{b-a}=z^{2\pi i(b-a)x}=1$ and we have

$$z^b=z^a=\alpha \text{ (or } \alpha^2). \tag{5.6}$$

Without loss of generality we can assume that $0<x<\frac{1}{2}$ and choose $r=\frac{1}{3}$ or $-\frac{1}{3}$ so that $e^{2\pi ibr}=e^{2\pi iar}=\alpha$. For $y:=x-r$, we have

$$e^{2\pi iay}=e^{2\pi iby}=1.$$

It follows that $ay=k_a$, $by=k_b$ for some integers k_a and k_b. Consequently $ak_b=bk_a$ and hence $b|k_b$. From $by=k_b$, we see that y is an integer. Since $|y|<1$, the only possibility is $y=0$, i.e., $x=\frac{1}{3}$ or $-\frac{1}{3}$ and $z=\alpha$ or α^2.

In the case when a is even, we use the alternative expression $Q(z)=1+z^{\frac{a}{2}}+z^b$. (5.4) becomes $z^{\frac{a}{2}}=\alpha$ (or α^2) and (5.5) is the same, which implies again that $a\equiv b$ (mod 3). To show that α and α^2 are the only roots of Q of modulus 1, we observe that $z^a=z^b$, $z^{\frac{a}{2}}=\alpha$ or α^2, so that (5.6) holds the same and the proposition follows. □

REFERENCES

[B] M. F. Barnsley, *Fractals Everywhere*, Second edition, Academic Press, Boston, 1993.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics 470, Springer-Verlag, Berlin-New York, 1975.

[CM] R. Cawley and R. D. Mauldin, *Multifractal decompositions of Moran fractals*, Adv. Math., 92 (1992), pp. 196–236.

[F] K. Falconer, *Techniques in Fractal Geometry*, Wiley, Chichester, 1997.

[FL] A.-H. Fan and K.-S. Lau, *Asymptotic behavior of multiperiodic functions* $G(x)=\prod_{n=1}^{\infty} g(x/2^n)$, J. Fourier Anal. Appl., 4 (1998), pp. 129–150.

[Fe1] D.-J. Feng, *The limit Rademacher functions and Bernoulli convolutions associated with Pisot numbers*, preprint, 1999.

[Fe2] ———, *On the infinite similarity and multifractal analysis of the Bernoulli convolutions*, preprint, 1999.

[G] A. M. Garsia, *Arithmetic properties of Bernoulli convolutions*, Trans. Amer. Math. Soc., 102 (1962), pp. 409–432.

[HL] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals I*, Math. Z., 27 (1928), pp. 565–606.

[H] L. Hervé, *Etude d'opérateurs quasi-compacts positifs. Applications aux opérateurs de transfert*, Ann. Inst. H. Poincaré Probab. Statist., 30 (1994), pp. 437–466.

[Hu] J. E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J., 30 (1981), pp. 713–747.

[JW] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc., 38 (1935), pp. 48–88.

[K] R. Kenyon, *Projecting the one-dimensional Sierpinski gasket*, Israel J. Math., 97 (1997), pp. 221–238.

[L1] K.-S. Lau, *Fractal measures and mean p-variations*, J. Funct. Anal., 108 (1992), pp. 427–457.

[L2] ———, *Dimension of a family of singular Bernoulli convolutions*, J. Funct. Anal., 116 (1993), pp. 335–358.

[LN1] K.-S. Lau and S.-M. Ngai, *Multifractal measures and a weak separation condition*, Adv. Math., 141 (1999), pp. 45–96.

[LN2] ———, *L^q-spectrum of the Bernoulli convolution associated with the golden ratio*, Studia Math., 131 (1998), pp. 225–251.

[LN3] ———, *L^q-spectrum of Bernoulli convolutions associated with P.V. numbers*, Osaka J. Math., 36 (1999), pp. 993–1010.

[LW] K.-S. Lau and J. Wang, *Mean quadratic variations and Fourier asymptotics of self-similar measures*, Monatsh. Math., 115 (1993), pp. 99–132.

[O] L. Olsen, *A multifractal formalism*, Adv. Math., 116 (1995), pp. 82–195.

[RW] H. Rao and Z.-Y. Wen, *A class of self-similar fractals with overlap structure*, Adv. in Appl. Math., 20 (1998), pp. 50–72.

[R] R. Riedi, *An improved multifractal formalism and self-similar measures*, J. Math. Anal. Appl., 189 (1995), pp. 462–490.

[Ru] D. Ruelle, *Thermodynamic Formalism. The Mathematical Structures of Classical Equilibrium Statistical Mechanics*, Encyclopedia of Mathematics and Its Applications 5, Addison-Wesley, Reading, Mass., 1978.

[S] R. Salem, *Algebraic Numbers and Fourier Analysis*, D. C. Heath and Co., Boston, Mass. 1963.

[St1] R. S. Strichartz, *Self-similar measures and their Fourier transforms I*, Indiana Univ. Math. J., 39 (1990), pp. 797–817.

[St2] ———, *Self-similar measures and their Fourier transforms II*, Trans. Amer. Math. Soc., 336 (1993), pp. 335–361.

[St3] ———, *Self-similar measures and their Fourier transforms III*, Indiana Univ. Math. J., 42 (1993), pp. 367–411.