Semigroup Forum Vol. 56 (1998) 151–156

RESEARCH ARTICLE

Embedding Locally Compact Semigroups into Groups

Ka-Sing Lau, Jimmie Lawson, and Wei-Bin Zeng

Communicated by D. R. Brown

1. Introduction

Let X, Y, Z be topological spaces. A function $F: X \times Y \to Z$ is called *jointly continuous* if it is continuous from $X \times Y$ with the product topology to Z. It is said to be *separately continuous* if $x \mapsto F(x,y): X \to Z$ is continuous for each $y \in Y$ and $y \mapsto F(x,y): Y \to Z$ is continuous for each $x \in X$. A *semitopological semigroup* is a semigroup S endowed with a topology such that the multiplication function is separately continuous, or equivalently, all left and right translations are continuous. In the case the semigroup is actually a group, we call it a semitopological group (it is often required that the inversion function be continuous also, but we omit this assumption). There is considerable literature (see [5] for a survey and bibliography) devoted to the problem of embedding a topological semigroup (a semigroup with jointly continuous multiplication) into a topological group, but the semitopological version appears to have received scant attention. On the other hand there are good reasons for considering this case, among them the fact that a number of general conditions exist for concluding that a semitopological group is actually a topological one. We further find that the semitopological setting gives much more straightforward statements and proofs of key results. The ideas of this paper parallel in many aspects those contained in the first part of [1]. Our Corollary 2.7 is essentially Theorem 2.1 of [1] extended from right reversible semigroups to general semigroups embedded in a group. Results paralleling most of the results of this paper can be found in Chapter VII of [4], except that we relax the hypothesis assumed on the semigroup S to only assuming translations are open mappings (a stronger condition on the semigroups S is assumed in [4] to guarantee continuity of inversion in the containing group). In addition, our methods here are much quicker and more direct.

The algebraic problem of giving necessary and sufficient conditions for group embeddability of a semigroup is a delicate one, although cancellativity is an obvious necessary condition. We bypass the algebraic problem by considering only semigroups that are (algebraically) group embeddable. If a semigroup S embeds in a group G, then there is a smallest subgroup in G containing S and we assume that our embeddings are always readjusted at the codomain level so that S generates (as a group) G.

2. Embedding Theorems

We maintain the terminology and notation of the introduction.

Theorem 2.1. *Let S be a semitopological semigroup which algebraically embeds into and generates a group G.*

A necessary and sufficient condition that there exist a (unique) topology on G making it a semitopological group such that S embeds homeomorphically onto an open subsemigroup of G is that all left and right translations in S be open mappings. In this case the group G is Hausdorff if and only if S is Hausdorff and the multiplication is jointly continuous on G if and only if it is on S.

Proof. Each translation in a semitopological group has a continuous inverse and hence is a homeomorphism. Thus the restriction to any open subset is an open mapping. This yields the necessity.

Conversely suppose that S is a semitopological semigroup in which all left and right translations are open mappings and that $\phi\colon S \to G$ is an algebraic embedding of S into the group G. We identify S with $\phi(S)$, regard S as a subset of G, and ϕ as the inclusion.

Let $x \in S$ be fixed, and let $\mathcal{N}_x$ denote the collection of all open subsets of S containg x.

For any $g \in G$, we define

$$\mathcal{B}_g := \{x^{-1}Ug\colon U \in \mathcal{N}_x\}.$$

We claim that for $O \in \mathcal{B}_y$ and $z \in O$, there exists $O' \in \mathcal{B}_z$ such that $O' \subseteq O$. Indeed, let $O = x^{-1}Uy$; then $z = x^{-1}uy$ for some $u \in U$. Since xu is in the open set xU, there exists and open set $U'' \in \mathcal{N}_x$ such that $U''u \subseteq xU$, i.e., $x^{-1}U''u \subseteq U$ in G. Since $U''x$ is an open set containing xx there exists an open set $U' \in \mathcal{B}_x$ such that $xU' \subseteq U''x$. Then $U'x^{-1} \subseteq x^{-1}U''$ in G. Let $O' := x^{-1}U'z \in \mathcal{B}_z$. We have

$$O' = x^{-1}U'x^{-1}uy \subseteq x^{-1}x^{-1}U''uy \subseteq x^{-1}Uy = O.$$

It now follows readily that $\bigcup_{g\in G}\mathcal{B}_g$ is a basis for a topology and that with respect to this topology $\mathcal{B}_g$ is a basis of open sets at g for each $g \in G$. Indeed if z is in the intersection $B_x \cap B_y$ for $B_x \in \mathcal{B}_x$ and $B_y \in \mathcal{B}_y$, then by the preceding paragraph we find $B_1, B_2 \in \mathcal{B}_z$ such that $B_1 \subseteq B_x$ and $B_2 \subseteq B_y$, and one verifies easily that $B_1 \cap B_2$ is again in $\mathcal{B}_z$. From these conclusions we can deduce immediately that all right translations by members of G are open mappings, and hence homeomorphisms (since they have open inverses).

There is a left-right dual to our preceding considerations, namely we could have defined a basis for a topology to consist of all sets of the form gUx^{-1}, where U is an open set in S containing x. Then the dual arguments show that these also form a basis of open sets for a topology on G in which left multiplications are homeomorphisms. Indeed we show that these two topologies coincide. To this end we claim that if $g \in G$ and $U \in \mathcal{N}_x$, then there exists $V \in \mathcal{N}_x$ such that $gVx^{-1} \subseteq x^{-1}Ug$. This establishes continuity between the two topologies in one direction and a dual argument gives the other. Noting that every member of G can be written as a finite product of members of $S \cup S^{-1}$ (since S generates G), we establish our claim by induction on the length of a product from this union.

Assume that the statement is true for all products of length less than n, and let sg be a product of length n, $s \in S$ and g of length $n-1$. Let $U \in \mathcal{N}_x$. Since $xsx \in Usx$, an open set, there exists

$U' \in \mathcal{N}_x$ such that $xsU' \subseteq Usx$, i.e., $sU'x^{-1} \subseteq x^{-1}Us$. Since $xx \in xU'$, an open set, there exists $W \in \mathcal{N}_x$ such that $Wx \subseteq xU'$, i.e., $x^{-1}W \subseteq U'x^{-1}$. By the inductive hypothesis there exists $V \in \mathcal{N}_x$ such that $gVx^{-1} \subseteq x^{-1}Wg$. Then

$$sgVx^{-1} \subseteq sx^{-1}Wg \subseteq sU'x^{-1}g \subseteq x^{-1}Usg.$$

Thus the two topologies agree and yield a semitopological group.

Finally we show that S homeomorphically embeds onto an open subset of G. Let $s \in S$ and let V be an open neighborhood of s in S.

Then xV is an open neighborhood of xs and so there exists $U \in \mathcal{N}_x$ such that $Us \subseteq xV$, i.e., $x^{-1}Us \subseteq V$ (in G). Since $x^{-1}Us$ is open in G, contains s, and is contained in $V \subseteq S$, we conclude that S is open in G and and the inclusion mapping $\phi: S \to G$ is open. Conversely given $U \in \mathcal{N}_x$, since Us is open in S, there exists V open containing s such that $xV \subseteq Us$. Then $V \subseteq x^{-1}Us$, which demonstrates that ϕ is also continuous.

Given any topology on G making it a semitopological group with S an open subsemigroup and ϕ a homeomorphism, then $\mathcal{N}_x$ will be a basis of open sets at x in G. Thus (since all translations are homeomorphisms) the sets $x^{-1}Ug$, $U \in \mathcal{N}_x$ will form a basis of open sets at $g \in G$. Hence the group topology on G must coincide with the one we have defined previously, i.e., such a topology is unique.

Clearly if G is Hausdorff, then the embedded semigroup S is also. Conversely suppose that S is Hausdorff. Since translations are homeomorphisms, to check that G is Hausdorff it suffices to take some fixed $x \in S$ and show that for any $g \in G$, $g \neq x$, there exist open neighborhoods of x and g that are disjoint. Since S is open, we may take the neighborhoods to be S and $x^{-1}Sg$ if these are disjoint. Otherwise there exists $s \in S$ such that $x^{-1}sg = t \in S$, and thus $sg = xt \in S^2 \subseteq S$. Now $x \neq g$ implies $sx \neq sg$. Since S is Hausdorff and open in G, we may find U, V open in S (and hence G) such that $sx \in U$, $sg \in V$ and $U \cap V = \emptyset$. Then $s^{-1}U$ and $s^{-1}V$ are disjoint neighborhoods of x and g respectively.

Finally we consider the setting of joint continuity together with the preceding hypotheses. If the multiplication on G is jointly continuous, then clearly that will be true for the restriction to (the embedded image of) S. Conversely it is a standard exercise in topological group theory that if multiplication is separately continuous everywhere in a group and jointly continuous at (e, e), then it is jointly continuous. We leave it as an exercise to show that one can substitute any (g, h) (particularly one in the open subset $S \times S$) for (e, e). ∎

Example 2.2. Let $(S, +) = ([0, \infty), +)$ with ordinary addition $+$ as the semigroup operation, and let S have the right half-open interval topology, with basis all intervals $[a, b) \subseteq S$. Then S is a Hausdorff topological semigroup in which all translations are open mappings. It embeds as open subsemigroup of the group of additive reals equipped with the right half-open interval topology. Note that although addition is jointly continuous, the additive reals with teh right half-open topology do not form a topological group since inversion is not continuous.

However, examples such as the preceding cannot occur in the locally compact setting.

Corollary 2.3. *Let S be a locally compact Hausdorff semitopological semigroup in which all translations are open mappings and which algebraically embeds into a group G, which it generates. Then G admits a unique locally compact Hausdorff topology making it a topological group such that S also topologically embeds as an open subsemigroup. In particular, the multiplication on S must be jointly continuous.*

Proof. By the preceding theorem we have that there exists a unique topology on G for which multiplication is separately continuous such that S also topologically embeds as an open subsemigroup. By the preceding theorem G is also Hausdorff. Since translations are homemorphisms and since G is locally compact at points of S, it is locally compact. We now apply results of Ellis [3] that assert that a locally compact Hausdorff group with separately continuous multiplication is a topological group. ∎

Definition 2.4. Let S be a cancellative Hausdorff semitopological semigroup.

A non-empty open subset U of S is said to be *translatable* if sU, Ut and sUt are again open sets for all $s,t \in S$. Let I_0 denote the union of all translatable open sets which have compact closures.

Lemma 2.5. *Let S be a cancellative Hausdorff semitopological semigroup which contains a translatable open subset U. Then I_0 is an open ideal and every non-empty open subset of I_0 is translatable.*

Proof. Since I_0 is a union of open sets, it is open. Let $x \in I_0$. Then there exists some translatable open set U such that $x \in U$. One easily verifies that sU and Ut are again translatable open sets for all $s,t \in S$; hence $sx, xt \in I_0$. Thus I_0 is a left and right ideal, hence an ideal.

Let W be a non-empty open subset of I_0. Let $x \in W$ and let $s \in S$. There exists some open translatable subset U with compact closure such that $x \in U$; set $V := U \cap W$. Let y_α be a net in S converging to sx. Since U is translatable, we have $y_\alpha \in sU$ for large enough α, and hence $y_\alpha = su_\alpha$, $u_\alpha \in U$ for large α. Since U has compact closure, some subnet $\{u_\beta\}$ of the net $\{u_\alpha\}$ converges to some $u \in \overline{U}$. Then $y_\beta = su_\beta \to su$, so $su = sx$ from the Hausdorff property. By cancellation $u = x$. So for large β, $u_\beta \in V$, and thus $y_\beta = su_\beta \in sV \subseteq sW$. We have thus shown that there does not exist a net outside of sW converging to sx, and hence sW must be a neighborhood of sx. Since x was arbitrary in W, we conclude that sW is open. Similar arguments establish that Wt and sWt are open for all $s,t \in S$. ∎

Theorem 2.6. *Let S be a Hausdorff semitopological semigroup which is algebraically embeddable in a group G (which it generates) and for which $I_0 \neq \emptyset$. Then G admits a unique topology for which it is a locally compact topological group such that I_0 topological embeds onto an open subset of G. The embedding is continuous on S and is a homeomorphic embedding if (left or right) translation by some member of I_0 is a homeomorphic embedding of S into I_0. If S is locally compact and A is any subset of S which topologically embeds as an open subset of G, then $A \subseteq I_0$.*

Proof. By the preceding lemma I_0 is an ideal, hence a subsemigroup. As usual, we identify S as a subset of G via the algebraic embedding $\phi : S \to G$. Let $s \in S$ and let $x \in I_0$. Then $sx \in I_0$, so $s = (sx)x^{-1}$ is in the subgroup of G generated by I_0. It follows that S and I_0 generate the same group, namely G. By the preceding lemma translations in I_0 are open mappings, and thus by Theorem applied to I_0, there exists a unique topology on G making it a Hausdorff semitopological group with I_0 topologically embedded as an open subset. Since $\phi(s) = \phi(sx)x^{-1}$ for some fixed $x \in I_0$ and the embedding restricted to I_0 is a homeomorphic embedding, we conclude that ϕ is continuous on S and is a homeomorphic embedding if (right) translation by x is a homeomorphism from S into S. Since each member x of I_0 has a compact neighborhood in S, the image of the neighborhood is compact in G and remains a neighborhood of x, since I_0 embeds as an open subset of G. Thus G is locally compact and hence (as in Corollary 2.3) a topological group.

Let A be an open subset of S that topological embeds as and open subset of G. Let $b \in A$ and let U be an open set containing b with compact closure contained in A. Then U is open in G since A is open in G and the embedding restricted to A is a topological embedding. Then for $s \in S$, sU is again open in G (since translations are homeomorphisms)and hence in S (since the embedding is continuous); similarly Ut and sUt are open subsets of S for all $s, t \in S$. We conclude that U is translatable with compact closure, and hence $b \in I_0$. Since b was arbitrary in A, we have $A \subseteq I_0$. ■

Corollary 2.7. *Let S be a Hausdorff semitopological semigroup which embeds algebraically into a group G (which it generates). Suppose further that each point of S contains a neighborhood which is homeomorphic to some subset of $\mathbf{R}^n$, euclidean n-space, and that the set S° of points having a a neighborhood homeomorphic to an open subset of $\mathbf{R}^n$ is non-empty. Then G admits a unique topology making it a Lie group such that S° embeds homeomorphically as an open subset of G. The embedding of S into G is continuous and is a homeomorphic embedding if (left or right) translation by some member of S° is a homeomophic embedding of S into S.*

Proof. Let $x \in S^\circ$. Then by passing to a small enough neighborhood, one finds a euclidean neighborhood U of x with compact closure. One can use the theorem of Invariance of Domain to establish that U is translatable (see, for example, the argument in the proof of Theorem 2.1 of [1]). One concludes that $S^\circ \subseteq I_0$. Applying Theorem 2.6, we conclude that S° topologically embeds as an open subset of G (since S° is open in S, hence in I_0).

Since S° is locally euclidean, so is G, and hence by the positive solution of Hilbert's fifth problem, G is a Lie group. ■

A semigroup S is said to be *right reversible* if $Sa \cap Sb \neq \emptyset$ for all $a, b \in S$. Note that commutative semigroups are always right reversible. It is standard (the Ore-Rees Theorem [2]) that a cancellative right reversible semigroup is algebraically embeddable into a group G such that $G = S^{-1}S$. Thus our results apply to cancellative right reversible (in particular, commutative) semigroups. We give one such typical application of Corollary 2.3.

Corollary 2.8. *Let S be a locally compact Hausdorff semitopological cancellative right reversible (in particular, commutative) semigroup in which all translations are open mappings. Let G denote the group of left quoients into which S embeds. Then G admits a unique locally compact Hausdorff topology making it a topological group such that S also topologically embeds as an open subsemigroup. In particular, the multiplication on S must be jointly continuous.*

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Department of Mathematics
Chinese University of Hong Kong
Hong Kong
China

Department of Mathematics
University of Louisville
Louisville, KY 40292

Department of Mathematics
Louisiana State University
Baton Rouge, LA 70808

Received May 27, 1990
and in final form April 14, 1998