EKELAND'S VARIATIONAL PRINCIPLE FOR SET-VALUED FUNCTIONS*

C. G. LIU † AND K. F. NG‡

Abstract. We establish several set-valued function versions of Ekeland's variational principle and hence provide some sufficient conditions ensuring the existence of error bounds for inequality systems defined by finitely many lower semicontinuous functions.

Key words. Ekeland's variational principle, partial order, error bound

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1. Introduction. The celebrated variational principle of Ekeland [13, 14] states that if f is a lower semicontinuous (lsc) bounded below function on a complete metric space (M, d) then for any $\gamma > 0$ and any x_0 with $f(x_0) < +\infty$ there exists $\bar{x} \in M$ such that

$$f(\bar{x}) \leq f(x_0) - \gamma d(x_0, \bar{x}) \tag{1.1}$$

and

$$f(\bar{x}) < f(x) + \gamma d(x, \bar{x}), \qquad \forall x \in M \backslash \{\bar{x}\}. \tag{1.2}$$

This principle is an important tool with a lot of significant applications in many areas including nonlinear analysis and optimization theory. There are many papers (see [6, 8, 11, 35] for example) reporting different formulations and some have put forward extended versions applicable to vector-valued/set-valued functions (see [3, 4, 5, 9, 15, 16, 18, 38]). This paper is devoted to further extensions which are especially relevant for the study of error bound issue for the following inequality system:

$$f_i(x) \leq 0, \qquad i \in \overline{1, n}, \tag{1.3}$$

where each f_i is a proper lsc function on a Banach space X. The system is said to have an error bound τ if there exists $\tau > 0$ such that

$$d(x, S) \leq \tau \sum_{i=1}^{n} f_i(x)_+, \qquad \forall x \in X, \tag{1.4}$$

where $f_i(x)_+ := \max\{f_i(x), 0\}$, and

$$S := \{x \in X : f_i(x) \leq 0, i \in \overline{1, n}\}. \tag{1.5}$$

Since the pioneering work of Hoffman [19], this notion (and its local version) have played an important role in many areas in mathematical programming and variational analysis (see the excellent surveys [23, 31] for results before 1997, and for more recent

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†Department of Mathematics, Jinan University, Guangzhou, 510632, P.R.China. (tcgliu@jnu.edu.cn)

‡Department of Mathematics (and IMS), The Chinese University of Hong Kong, Shatin, New Territory, Hong Kong. (kfng@math.cuhk.edu.hk)

results see [1, 2, 7, 10, 12, 27, 28, 29, 30, 36] and references therein). The earlier results all are either under certain convexity assumption or restricted to the special case when $n = 1$. The results obtained in Section 4 are for the general case that n can be any positive integer, and provide sufficient conditions ensuring that the inequality system (1.3) has an error bound. They are established via set-valued versions of Ekeland's variational principle obtained in Section 3.

2. Notations and Preliminary Results. In general we use (M, d) to denote a metric space while X, Y and Z usually denote normed spaces (or Banach spaces); $B(x, r)$ denotes the closed ball with center x and radius $r > 0$. For short B_Z denotes the closed unit ball in Z and $S_Z := \{z \in Z : \|z\| = 1\}$ is the unit sphere. Let $\langle z^*, z\rangle := z^*(z)$. Given any subset K in Z, $S(K) = K \cap S_Z$. Let $cl(K)$, $co(K)$, $\overline{co}(K)$, $coneK$ and $\overline{cone}K$ respectively denote the closure, convex hull, closed convex hull, (convex) conic hull, and closed (convex) conic hull of K. Let $d(\cdot, K)$ denote the distance function of K, i.e.,

$$d(z, K) := \inf\{\|z - y\| : y \in K\}.$$

We say that a vector space Y is ordered by a convex cone $C \subset Y$, if Y is equipped with a binary relation (quasiorder) $\leq_C$ for elements in Y such that

$$y_1 \leq_C y_2 \Longleftrightarrow y_2 - y_1 \in C \tag{2.1}$$

($\leq_C$ is a partial order if and only if C is pointed, i.e., $C \cap (-C) = \{0\}$). For example, in multi-objective optimization problems, we often let $\mathbb{R}^n$ be ordered by $\mathbb{R}^n_+$, where $\mathbb{R}^n_+$ consists of all n-vectors $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$ such that each $y_i \geq 0$. Here the norm for $\mathbb{R}^n$ can be the usual Euclidean norm $\|y\| = \left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}$ or the l_1-norm $\|y\|_1 = \sum_{i=1}^n |y_i|$. One of the advantages of using the l_1-norm is that,

$$\text{if } H := \{y \in \mathbb{R}^n_+ : \|y\|_1 = 1\} \text{ then } d(0, H) = 1 \tag{2.2}$$

(and H is closed and convex). For convenience, we henceforth use the l_1-norm for $\mathbb{R}^n$.

As usual for a set-valued function $F : M \to 2^Y$, we use $dom\, F$ and $graph\, F$ to denote the domain and the graph of F respectively, that is, $dom\, F := \{x \in M : F(x) \neq \emptyset\}$, $graph\, F := \{(x, y) \in M \times Y : y \in F(x)\}$. Moreover, if Y is ordered by C, $epi\, F := \{(x, y) \in M \times Y : y \in F(x) + C\}$ is the epigraph of F. Thus

$$v \leq_C y, v \in F(x) \Longrightarrow (x, y) \in epi\, F. \tag{2.3}$$

For any two nontrivial closed convex cones K_1, K_2 in a normed space Y we use $\measuredangle(K_1, K_2)$ to denote the quantity (cf. [24, equation (2.2)])

$$\measuredangle(K_1, K_2) := \inf\{d(k, K_2) : k \in S(K_1)\}. \tag{2.4}$$

Lemma 2.1. *Let Y be a normed space ordered by a convex cone $C \subset Y$, and let C_0 be a convex cone such that $\{0\} \neq C_0 \subset C$. Then the following implication is valid for all $y_1 \in C_0$ and $y_2 \in Y$:*

$$y_1 \leq_C y_2 \Longrightarrow y_2 \in C \text{ and } \measuredangle(C_0, -C)\|y_1\| \leq \|y_2\|. \tag{2.5}$$

Proof. By (2.4), we have

$$\measuredangle(C_0, -C) \cdot \|y\| \leq d(y, -C), \qquad \forall y \in C_0. \tag{2.6}$$

Let $y_1 \in C_0$ and $y_2 \in Y$ with $y_1 \leq_C y_2$. Then $y_2 \in y_1 + C \subset C$. Moreover since $y_1 - y_2 \in -C$, one has by (2.6),

$$\measuredangle(C_0, -C) \cdot \|y_1\| \leq d(y_1, -C) \leq \|y_1 - (y_1 - y_2)\| = \|y_2\|.$$

This proves (2.5). □

REMARK 2.1. *It is well-known (see [32]) that C is normal in $(Y, \|\cdot\|)$ (in the sense that any $\{y_n\}_{n\in\mathbb{N}}$ converges to zero whenever there are sequences $\{x_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}}$ convergent to zero such that $x_n \leq_C y_n \leq_C z_n$ for each n) if and only if there is an equivalent norm $\|\cdot\|_1$ on Y such that*

$$y_1 \leq_C y_2 \Longrightarrow \|y_1\|_1 \leq \|y_2\|_1, \qquad \forall y_1, y_2 \in C. \tag{2.7}$$

In view of (2.4) and Lemma 2.1, it follows immediate that C is normal if and only if

$$\measuredangle(C, -C) > 0. \tag{2.8}$$

In particular, if (2.7) is satisfied with $\|\cdot\|$ in place of $\|\cdot\|_1$ (this condition is automatically satisfied if Y is a Banach lattice with positive cone C; see [34, Definition II.1.2, II.5.1]). Then $\measuredangle(C_0, -C) = 1$ for any convex cone C_0 such that $\{0\} \neq C_0 \subset C$. Indeed, if $y_1 \in S(C_0)$ then $d(y_1, -C) = \inf\{\|y_1 + c\| : c \in C\} = \|y_1\| = 1$.

EXAMPLE 2.1. *Let Y be a Hilbert space. Fix $e \in S(Y)$ and let $C := \{y \in Y : \langle e, y\rangle \geq \frac{1}{\sqrt{2}}\|y\|\}$. It can be verified that $\measuredangle(C_0, -C) = 1$ for any convex cone C_0 with $\{0\} \neq C_0 \subset C$.*

Let A be a nonempty subset of a metric space (M, d), and let $\preceq$ be a partial order defined on A. Recall that a point $\bar{a} \in A$ is called a minimal point of A if there does not exist $a \in A\backslash\{\bar{a}\}$ such that $a \preceq \bar{a}$. The set of all the minimal points of A is denoted by $Min(A, \preceq)$. Recall that A is said to have the domination property with respect to $\preceq$ if for each $x_0 \in A$, there is $\bar{x} \in A$ such that

$$\bar{x} \preceq x_0 \quad \text{and} \quad \bar{x} \in Min(A, \preceq). \tag{2.9}$$

The next result is due to Hamel and Tammer and would be convenient to be stated in the following form:

LEMMA 2.2. *Let A be a nonempty subset of a metric space (M, d), and let $\preceq$ be a partial order defined on A such that any decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in A converges to some $\bar{a} \in A$ with*

$$\bar{a} \preceq x_n \qquad \forall n \in \mathbb{N}. \tag{2.10}$$

Then A has the domination property with respect to $\preceq$.

Proof. This follows immediately from [18, Theorem 2.2]. □

3. Partial Orders Generated by Set-Valued Functions. Let (M, d) be a complete metric space and let Y be a Banach space ordered by a nontrivial closed convex cone C. Let $F : M \to 2^Y$ be a set-valued function.

DEFINITION 3.1. *Let $\gamma > 0$ and let $H \subset C\backslash\{0\}$ be a closed convex set such that*

$$\measuredangle(\overline{cone} H, -C) > 0. \tag{3.1}$$

We define relations $\preceq_{(F,\gamma,H)}$ and $\precsim_{(F,\gamma,H)}$ (or $\preceq, \precsim$ for short if no confusion can arise) on $dom\, F$ by

$$\begin{aligned} &x_1 \preceq_{(F,\gamma,H)} x_2 \Longleftrightarrow \\ &\quad \forall y_2 \in F(x_2), \exists y_1 \in F(x_1), h \in H, \ s.t. \ \gamma d(x_1, x_2) h \leq_C y_2 - y_1, \end{aligned} \tag{3.2}$$

and

$$x_1 \precsim_{(F,\gamma,H)} x_2 \Longleftrightarrow \forall \gamma' \in (0,\gamma),\ x_1 \preceq_{(F,\gamma',H)} x_2 \tag{3.3}$$

respectively, where $x_1, x_2 \in dom\, F$.

By (3.2), we have the following equivalences:

$$x_1 \preceq_{(F,\gamma,H)} x_2 \Longleftrightarrow F(x_2) \subset F(x_1) + \gamma d(x_1,x_2)H + C \tag{3.4}$$

$$\Longleftrightarrow \forall y_2 \in F(x_2), \exists h \in H \text{ s.t. } (x_1, y_2 - \gamma d(x_1,x_2)h) \in epi\, F. \tag{3.5}$$

Since $H \subset C$ and by (2.3), the following implications are also valid:

$$x_1 \preceq_{(F,\gamma,H)} x_2 \Longrightarrow F(x_2) \subset F(x_1) + C, \tag{3.6}$$

$$x_1 \preceq_{(F,\gamma,H)} x_2 \Longrightarrow (x_1, y_2) \in epi\, F,\ \forall y_2 \in F(x_2). \tag{3.7}$$

It is clear that for any $x_1, x_2 \in dom\, F$,

$$x_1 \preceq_{(F,\gamma,H)} x_2 \Longrightarrow x_1 \precsim_{(F,\gamma,H)} x_2. \tag{3.8}$$

But generally, the inverse does not hold.

Example 3.1. *Let $Y = l_2$, the Hilbert space consisting of all square-summable sequences of real numbers. For each $n \in \mathbb{N}$, let e_n denote the element in l_2 whose n^{th} coordinate is 1 and other coordinates are zero. Let $C := \{y \in l_2 : \langle e_1, y\rangle \geq \frac{1}{\sqrt{2}}\|y\|\}$. Consider two distinct points x_1, x_2 in metric space (M,d) with*

$$d(x_1,x_2) = 1. \tag{3.9}$$

Let $H := \{y \in C : \langle e_1, y\rangle = 1\}$. Then H is a closed convex subset of C such that $coneH = C$ (so $coneH$ is closed). By Example 2.1,

$$\measuredangle(\overline{cone}H, -C) = 1. \tag{3.10}$$

Let $F : \{x_1, x_2\} \to 2^Y$ be defined by

$$F(x_2) = \{0\}, \tag{3.11}$$

$$F(x_1) = \{(\frac{1}{n} - 1)(e_1 + e_n)\}_{n\geq 2}. \tag{3.12}$$

We claim that

$$x_1 \precsim_{(F,1,H)} x_2, \tag{3.13}$$

but that

$$x_1 \not\preceq_{(F,1,H)} x_2, \tag{3.14}$$

First, since $e_1 + e_n \in H$ for all $n \geq 2$, we have

$$F(x_2) \subset F(x_1) + (1 - \frac{1}{n})H + C, \qquad \forall n \geq 2. \tag{3.15}$$

Also, for any $\gamma' \in (0,1)$, there exists $n' \geq 2$ such that $\gamma' < 1 - \frac{1}{n'}$. Hence by (3.15) *and the fact that $(1 - \frac{1}{n'})H \subset \gamma' H + C$, it follows from* (3.9) *that*

$$\begin{aligned} F(x_2) &\subset F(x_1) + \gamma' H + C \\ &= F(x_1) + \gamma' d(x_1,x_2)H + C. \end{aligned} \tag{3.16}$$

Together with (3.3) *and* (3.4)*, we obtain* (3.13)*.*

Second, since $(\frac{1}{n}-1)(e_1+e_n) \notin -H-C$ *for all* $n \geq 2$*, we have*

$$F(x_1) \cap (-H-C) = \emptyset, \tag{3.17}$$

and then

$$F(x_2) = \{0\} \nsubseteq F(x_1) + H + C. \tag{3.18}$$

Together with (3.4)*,* (3.14) *holds.*

Recall that a subset D of Y is said to be C-bounded (cf. [25, pp. 13-14]) if there exists some bounded set $D_0 \subset Y$ such that $D \subset D_0 + C$. For example, with $(Y, C) = (\mathbb{R}, \mathbb{R}_+)$, a subset D of $\mathbb{R}$ is $\mathbb{R}_+$-bounded if and only if D is a bounded below set of real numbers.

For the remainder of this section, the following assumptions will be considered:

(A1) $H \subset C \backslash \{0\}$ is a closed convex set (thus we have $\kappa := d(0, H) > 0$).

(A2) $\zeta := \measuredangle(\overline{cone} H, -C) > 0$.

(A3) $F(M)$ is C-bounded: there exists $\eta > 0$ such that

$$\forall y \in F(M), \exists u \in Y \text{ s.t. } \|u\| \leq \eta \text{ and } u \leq_C y. \tag{3.19}$$

(A4) $epi\, F$ is closed in $M \times Y$.

EXAMPLE 3.2. *In the ordered Banach space* $(Y, C) = (\mathbb{R}^n, \mathbb{R}^n_+)$*, let* H *be defined as* (2.2)*. Then (A1) is satisfied. Moveover, it is obvious that* $\overline{cone} H = \mathbb{R}^n_+$ *and*

$$\measuredangle(\overline{cone} H, -\mathbb{R}^n_+) = 1. \tag{3.20}$$

So (A2) is also true.

For the inequality system (1.3)*, let* $D := \cap_{i=1}^n dom\, f_i$ *and let* $F : X \to \mathbb{R}^n$ *be a set-valued function defined as*

$$F(x) := \begin{cases} \{(f_1(x)_+, f_2(x)_+, \cdots, f_n(x)_+)\} & x \in D \\ \emptyset & x \notin D \end{cases}. \tag{3.21}$$

Then $F(X) \subset \mathbb{R}^n_+$ *is* $\mathbb{R}^n_+$*-bounded. It is easy to verify that* $epi\, F$ *is closed. Thus (A3) and (A4) are satisfied.*

The following proposition provides a sufficient condition ensuring that $\preceq$ and $\precsim$ are partial orders:

PROPOSITION 3.2. *Consider* $\gamma > 0$ *and* F, H *satisfying assumptions (A1)-(A3) with the associated constants* $\kappa, \zeta, \eta > 0$*. Then, both the relations* $\preceq_{(F,\gamma,H)}$ *and* $\precsim_{(F,\gamma,H)}$ *defined in Definition 3.1 are partial orders on* $dom\, F$*.*

Proof. We need only to show that the relation $\preceq$ is a partial order on $dom\, F$ (as the corresponding result for $\precsim$ follows easily). It is easy to see that the relation $\preceq$ is reflexive, that is, $x \preceq x$ for all $x \in dom\, F$. Let x_1, x_2, x_3 be distinct elements of $dom\, F$ such that $x_1 \preceq x_2$ and $x_2 \preceq x_3$. Then for any $y_3 \in F(x_3)$, there exist $y_2 \in F(x_2)$, $y_1 \in F(x_1)$ and $h_1, h_2 \in H$ such that $\gamma d(x_1, x_2) h_1 \leq_C y_2 - y_1$ and $\gamma d(x_2, x_3) h_2 \leq_C y_3 - y_2$. Let $h_3 := \frac{d(x_1,x_2)}{d(x_1,x_2)+d(x_2,x_3)} h_1 + \frac{d(x_2,x_3)}{d(x_1,x_2)+d(x_2,x_3)} h_2$. Then $h_3 \in H \subset C$ and

$$\begin{aligned} \gamma d(x_1, x_3) h_3 &\leq_C \gamma \left[d(x_1, x_2) + d(x_2, x_3) \right] h_3 \\ &\leq_C y_3 - y_1. \end{aligned} \tag{3.22}$$

So $x_1 \preceq x_3$. To prove the anti-symmetry of $\preceq$, suppose that $x \preceq x'$ and $x' \preceq x$. By what has just been proved, for any $z \in F(x)$, there exist $z' \in F(x)$ and $h' \in H$ such that

$$2\gamma d(x,x')h' \leq_C z - z'. \tag{3.23}$$

Therefore, inductively, there exist sequences $\{z_n\}_{n\in\mathbb{N}} \subset F(x)$ and $\{h'_n\}_{n\in\mathbb{N}} \subset H$ such that

$$2\gamma d(x,x')h'_n \leq_C z_n - z_{n+1}, \qquad \forall n \in \mathbb{N}. \tag{3.24}$$

By assumption (A3), there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset Z$ such that

$$\|u_n\| \leq \eta, \qquad \forall n \in \mathbb{N}, \tag{3.25}$$

and

$$u_n \leq_C z_n, \qquad \forall n \in \mathbb{N}. \tag{3.26}$$

It follows from (3.24) and (3.26) that

$$2\gamma d(x,x')\sum_{i=1}^{n} h'_i \leq_C z_1 - z_{n+1} \leq_C z_1 - u_{n+1}, \qquad \forall n \in \mathbb{N}. \tag{3.27}$$

Since $H \subset C\backslash\{0\}$ and C is a closed convex cone, $C_0 := \overline{cone} H \subset C$. Using (2.5) with y_1 and y_2 replaced by $2\gamma d(x,x')\sum_{i=1}^{n} h'_i$ and $z_1 - u_{n+1}$, we have

$$\begin{aligned}\zeta \cdot 2\gamma d(x,x') \cdot n\kappa &\leq \zeta \cdot \left\| 2\gamma d(x,x')\sum_{i=1}^{n} h'_i \right\| \\ &\leq \|z_1 - u_{n+1}\| \\ &\leq (\|z_1\| + \eta), \qquad \forall n \in \mathbb{N}.\end{aligned} \tag{3.28}$$

Since κ, ζ, η and γ are positive constants it follows that $d(x,x') = 0$ and so $x = x'$. ◻

Remark 3.1. *Assumption (A3) in Proposition 3.2 can be relaxed to the condition that $F(x)$ is C-bounded for each $x \in dom\, F$.*

Lemma 3.3. *Consider $\gamma > 0$ and F, H satisfying assumptions (A1)-(A4) with the associated constants $\kappa, \zeta, \eta > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of $dom\, F$ with respect to $\preceq_{(F,\gamma,H)}$ ($\preceq$ for short). Then $\{x_n\}_{n\in\mathbb{N}}$ converges to some $\bar{a} \in dom\, F$, and*

$$F(x_n) \subset F(\bar{a}) + C, \qquad \forall n \in \mathbb{N}. \tag{3.29}$$

The same assertion is also true for $\precsim_{(F,\gamma,H)}$ in place of $\preceq_{(F,\gamma,H)}$.

Proof. As the last assertion follows easily from the first, we only need to prove the results regarding $\preceq$. First, we show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Suppose not, without loss of generality, we may assume that there exists $\varepsilon > 0$ such that

$$d(x_n, x_{n+1}) > \epsilon, \qquad \forall n \in \mathbb{N}. \tag{3.30}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is decreasing, there are sequences $\{y_n\}_{n\in\mathbb{N}}$, and $\{h_n\}_{n\in\mathbb{N}} \subset H$ such that

$$y_n \in F(x_n), \qquad \forall n \in \mathbb{N}, \tag{3.31}$$

$$\gamma d(x_n, x_{n+1})h_n \leq_C y_n - y_{n+1}, \qquad \forall n \in \mathbb{N}. \tag{3.32}$$

By (3.19), there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset Y$ such that

$$\|u_n\| \le \eta \text{ and } u_n \le_C y_n. \tag{3.33}$$

It follows from (3.32) that

$$\begin{aligned} y_1 - u_{n+1} \ge_C y_1 - y_{n+1} &= \sum_{i=1}^{n} (y_i - y_{i+1}) \\ &\ge_C \sum_{i=1}^{n} \gamma d(x_i, x_{i+1}) h_i \\ &= \sum_{i=1}^{n} \gamma \alpha_i h_i = \gamma\beta_n \left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta_n} h_i \right), \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{3.34}$$

where $\alpha_i := d(x_i, x_{i+1})$ and $\beta_n := \sum_{i=1}^{n} \alpha_i$. Noting by (3.30) that $\beta_n > n\epsilon$, and it follows from (2.5), (3.33) together with the assumed (A1) that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|y_1\| + \eta &\ge \|y_1 - u_{n+1}\| \\ &\ge \zeta \cdot \gamma\beta_n \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\beta_n} h_i \right\| \ge \zeta \cdot \gamma n\epsilon \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\beta_n} h_i \right\| \ge \zeta \cdot \gamma n\epsilon \cdot \kappa. \end{aligned} \tag{3.35}$$

But this is impossible because κ, ζ, η and γ are positive constants while n is arbitrary. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and hence converges to some $\bar{a} \in M$.

Moreover, note that for any fixed $n \in \mathbb{N}$, $\{x_k\}_{k\ge n}$ also converges to $\bar{a}$; this together with (3.2) and (3.6) implies that for any $y \in F(x_n)$, $\{(x_k, y) : k \ge n\}$ is a sequence in *epi* F and converges to $(\bar{a}, y)$. Hence $(\bar{a}, y) \in$ *epi* F thanks to the assumption that *epi* F is closed. Therefore $\bar{x} \in$ *dom* F and $y \in F(\bar{a}) + C$ for each $y \in F(x_n)$, that is, (3.29) is true. □

Next we present a set-valued version of Ekeland's variational principle type.

THEOREM 3.4. *Suppose all the assumptions in Lemma 3.3 are satisfied. Then* *dom* F *has the domination property with respect to* $\precsim_{(F,\gamma,H)}$, *namely, for each* $x_0 \in$ *dom* F, *there is* $\bar{x} \in$ *dom* F *such that*

$$\bar{x} \precsim_{(F,\gamma,H)} x_0 \quad \textit{and} \quad \bar{x} \in Min(dom\, F, \precsim_{(F,\gamma,H)}); \tag{3.36}$$

in other words,

$$F(x_0) \subset \cap_{\gamma' \in (0,\gamma)} \left[F(\bar{x}) + \gamma' d(x_0, \bar{x}) H + C \right], \tag{3.37}$$

and

$$F(\bar{x}) \not\subseteq \cap_{\gamma' \in (0,\gamma)} \left[F(x) + \gamma' d(x, \bar{x}) H + C \right], \quad \forall x \in (dom\, F) \backslash \{\bar{x}\}. \tag{3.38}$$

Proof. We use $\precsim$ to denote $\precsim_{(F,\gamma,H)}$ for simplicity. By Proposition 3.2, $\precsim$ is a partial order on *dom* F. Let $\{x_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of *dom* F with respect to $\preceq$. By Lemma 3.3, $\{x_n\}_{n\in\mathbb{N}}$ converges to some $\bar{a} \in$ *dom* F and (3.29) holds. By Lemma 2.2, we need only to show that

$$\bar{a} \precsim x_n, \qquad \forall n \in \mathbb{N}. \tag{3.39}$$

Let $n \in \mathbb{N}$ be fixed and consider any $\gamma' \in (0, \gamma)$. Since $\{x_k\}_{k \geq n}$ converges to $\bar{a}$, there exists $k' \geq n$ such that

$$d(x_{k'}, \bar{a}) \leq \frac{\gamma - \gamma'}{\gamma + \gamma'} d(x_n, \bar{a}). \tag{3.40}$$

Noting that $x_{k'} \precsim x_n$ and $\frac{\gamma+\gamma'}{2} \in (0, \gamma)$, it follows that for any $y \in F(x_n)$, there exist $y' \in F(x_{k'})$ and $h' \in H$ such that

$$\frac{\gamma + \gamma'}{2} d(x_n, x_{k'}) h' \leq_C y - y'. \tag{3.41}$$

By (3.29), there exists $\bar{y} \in F(\bar{a})$ such that

$$\bar{y} \leq_C y'. \tag{3.42}$$

And by (3.40), we have

$$d(x_n, x_{k'}) \geq d(x_n, \bar{a}) - d(x_{k'}, \bar{a}) \geq \frac{2\gamma'}{\gamma + \gamma'} d(x_n, \bar{a}). \tag{3.43}$$

Hence, by (3.41), (3.42) and (3.43), we have

$$\gamma' d(x_n, \bar{a}) h' \leq_C y - \bar{y}. \tag{3.44}$$

Thus $\bar{a} \precsim x_n$ by (3.3). So (3.39) holds and we complete the proof. □

REMARK 3.2. *Ekeland's variational principle (that we stated at the beginning) can be deduced immediately from Theorem 3.4 and Example 3.2 (with* $n = 1$ *and* $f_1(x) = f(x) - \inf_M f(\cdot)$*) and the fact that*

$$\cap_{\gamma' \in (0,\gamma)} [\{f_1(\bar{x})\} + \gamma' d(x_0, \bar{x}) + [0, +\infty)] = \{f_1(\bar{x})\} + \gamma d(x_0, \bar{x}) + [0, +\infty) \tag{3.45}$$

and

$$\cap_{\gamma' \in (0,\gamma)} [\{f_1(x)\} + \gamma d(x, \bar{x}) + [0, +\infty)] = \{f_1(x)\} + \gamma d(x, \bar{x}) + [0, +\infty). \tag{3.46}$$

Next we discuss the domination property of *dom* F with respect to $\preceq_{(F,\gamma,H)}$. The following theorem is similar to (but distinct from) Theorem 3.4. In fact Example 3.1 has shown that $\preceq_{(F,\gamma,H)}$ and $\precsim_{(F,\gamma,H)}$ are distinct, even when *dom* F has the domination property with respect to each of the two relations. Also there are examples to show that Theorem 3.5 would not longer be valid if (i), (ii) and (iii) are dropped.

Recall that the positive polar of C is defined as

$$C^+ := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall z \in C\}.$$

THEOREM 3.5. *Suppose all the assumptions in Lemma 3.3 are satisfied, and that (at least) one of the following assertions holds:*

(i) There exist $y_0^* \in S(C^+)$ *and* $\xi \in (0, 1)$ *such that*

$$H \subset \{y \in Y : \xi \|y\| \leq \langle y_0^*, y \rangle\}. \tag{3.47}$$

(ii) H *is bounded.*

(iii) Y is reflexive.
Then $dom\, F$ has the domination property with respect to $\preceq_{(F,\gamma,H)}$, namely, for each $x_0 \in dom\, F$, there is $\bar{x} \in dom\, F$ such that

$$\bar{x} \preceq_{(F,\gamma,H)} x_0 \quad and \quad \bar{x} \in Min(dom\, F, \preceq_{(F,\gamma,H)}); \tag{3.48}$$

in other words,

$$F(x_0) \subset F(\bar{x}) + \gamma d(x_0, \bar{x})H + C, \tag{3.49}$$

and

$$F(\bar{x}) \not\subseteq F(x) + \gamma d(x, \bar{x})H + C, \qquad \forall x \in (dom\, F)\backslash\{\bar{x}\}. \tag{3.50}$$

Proof. We use $\preceq$ to denote $\preceq_{(F,\gamma,H)}$ for simplicity. By Proposition 3.2, $\preceq$ is a partial order on $dom\, F$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of $dom\, F$ with respect to $\preceq$. By Lemma 3.3, $\{x_n\}_{n\in\mathbb{N}}$ converges to some $\bar{a} \in dom\, F$ and (3.29) holds. Similar to Theorem 3.4, we need only to show that

$$\bar{a} \preceq x_n, \qquad \forall n \in \mathbb{N}. \tag{3.51}$$

Let $n_0 \in \mathbb{N}$ be fixed. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to $\bar{a}$, there exists a subsequence $\{z_n\}_{n\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$ such that

$$z_1 = x_{n_0}, \tag{3.52}$$

and

$$z_{n+1} \preceq z_n \text{ and } d(z_{n+1}, \bar{a}) < \frac{1}{n+1}, \qquad \forall n \in \mathbb{N}. \tag{3.53}$$

For any fixed $y_1 \in F(z_1)$, let $\{y_n\}_{n\in\mathbb{N}} \subset Y$, $\{h'_n\}_{n\in\mathbb{N}} \subset H$ and $\{u_n\}_{n\in\mathbb{N}} \subset Y$ such that

$$y_n \in F(z_n), \qquad \forall n \in \mathbb{N}, \tag{3.54}$$

$$\gamma d(z_n, z_{n+1})h'_n \leq_C y_n - y_{n+1}, \qquad \forall n \in \mathbb{N}, \tag{3.55}$$

$$\|u_n\| \leq \eta \text{ and } u_n \leq_C y_n, \qquad \forall n \in \mathbb{N}. \tag{3.56}$$

It follows that

$$\begin{aligned} & y_1 - u_{n+1} \geq_C y_1 - y_{n+1} \\ & \quad \geq_C \sum_{i=1}^{n} [\gamma d(z_i, z_{i+1})h'_i] = \gamma\beta_n \left[\sum_{i=1}^{n} \frac{\alpha_i}{\beta_n} h'_i\right] = \gamma\beta_n h''_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.57}$$

where $\alpha_i := d(z_i, z_{i+1})$, $\beta_n := \sum_{i=1}^{n} \alpha_i$ and $h''_n := \sum_{i=1}^{n} \frac{\alpha_i}{\beta_n} h'_i \in H$. Using (2.5) together with the properties of η, ζ and κ given in (A1)-(A4), we get

$$\|y_1\| + \eta \geq \zeta \cdot \gamma\beta_n \|h''_n\| \geq \zeta \cdot \gamma\beta_n \cdot \kappa, \qquad \forall n \in \mathbb{N}. \tag{3.58}$$

This implies that $\{\beta_n\}_{n\in\mathbb{N}}$ is bounded and hence converges to a finite limit, say d_0, that is, $d_0 = \sum_{j=1}^{+\infty} d(z_j, z_{j+1}) < +\infty$. Since $\beta_n = \sum_{j=1}^{n} d(z_j, z_{j+1}) \geq d(z_1, \bar{a}) - d(z_{n+1}, \bar{a}) > d(z_1, \bar{a}) - \frac{1}{n+1}$ for all $n \in \mathbb{N}$, it follow from (3.57) that, for each n,

$$y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1}\right] h''_n \geq_C y_1 - \gamma\beta_n h''_n \geq_C y_{n+1}, \tag{3.59}$$

that is,

$$y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1} \right] h''_n \in y_{n+1} + C, \qquad \forall n \in \mathbb{N}. \tag{3.60}$$

Since $y_{n+1} \in F(z_{n+1}) \subset F(\bar{a}) + C$ (by (3.54) and (3.29) applied to z_{n+1} in place of x_n), it follows (3.60) that

$$y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1} \right] h''_n \in F(\bar{a}) + C \tag{3.61}$$

and so

$$\left(\bar{a}, y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1} \right] h''_n \right) \in epi\, F, \qquad \forall n \in \mathbb{N}. \tag{3.62}$$

We now split the proof into the following three cases.

(i) **There exists $y_0^* \in S(C^+)$ and $\xi \in (0,1)$ such that** (3.47) **holds.** It follows from (3.56), (3.57) and (3.47) that

$$\|y_1\| + \eta \geq \langle y_0^*, y_1 - u_{n+1} \rangle \geq \langle y_0^*, \gamma \sum_{i=1}^{n} \alpha_i h'_i \rangle \geq \gamma \xi \sum_{i=1}^{n} \alpha_i \|h'_i\|, \tag{3.63}$$

for all n. This implies that $\sum_{i=1}^{+\infty} \alpha_i \|h'_i\|$ is a convergent series. Moreover, for all $m, n \in \mathbb{N}$ with $m < n$, we have

$$h''_n - h''_m = \sum_{i=1}^{m} \alpha_i h'_i \left(\frac{1}{\beta_n} - \frac{1}{\beta_m} \right) + \sum_{i=m+1}^{n} \frac{\alpha_i h'_i}{\beta_n}.$$

Hence

$$\begin{aligned} \|h''_n - h''_m\| &\leq \frac{\beta_n - \beta_m}{\beta_n \cdot \beta_m} \left(\sum_{i=1}^{m} \alpha_i \|h'_i\| \right) + \frac{1}{\beta_n} \left(\sum_{i=m+1}^{n} \alpha_i \|h'_i\| \right) \\ &\leq \frac{\beta_n - \beta_m}{\beta_n \cdot \beta_m} (\gamma\xi)^{-1} (\|y_1\| + \eta) + \frac{1}{\beta_n} \left(\sum_{i=m+1}^{n} \alpha_i \|h'_i\| \right). \end{aligned} \tag{3.64}$$

Since β_m, β_n converge to d_0 and $\sum_{i=m+1}^{n} \alpha_i \|h'_i\|$ converges to 0 when $m \to \infty$, the sequence $\{h''_n\}$ is Cauchy. Let $h_0 := \lim_{n\to\infty} h''_n$; then $h_0 \in H$, by (3.62) we get

$$(\bar{a}, y_1 - \gamma d(z_1, \bar{a}) h_0) \in epi\, F. \tag{3.65}$$

Since y_1 is arbitrary in $F(z_1)$ and $z_1 = x$, this implies that (3.51) holds (see (3.5)).

(ii) H **is bounded, namely** $\sup_H \|\cdot\| < +\infty$**.** For any $v \in H + C$, there exist $v_1 \in H$ such that $v_1 \leq_C v$. Then, by (2.5), we have

$$\|v\| \geq \|v_1\| \cdot \zeta \geq \kappa \cdot \zeta,$$

and so

$$\inf_{H+C} \|\cdot\| \geq \kappa \cdot \zeta > 0. \tag{3.66}$$

This implies that $cl(H+C)$ (which is a closed convex set) does not contain 0. By the separation theorem, there exists $y_0^* \in Y^*$ with $\|y_0^*\| = 1$ such that

$$\inf_{cl(H+C)} \langle y_0^*, \cdot \rangle > 0. \tag{3.67}$$

Thus $y_0^* \in S(C^+)$ and

$$\inf_H \langle y_0^*, \cdot \rangle > 0 \tag{3.68}$$

Hence one can pick $\xi \in (0,1)$ such that

$$\xi < \frac{\inf_H \langle y_0^*, \cdot \rangle}{\sup_H \|\cdot\|}. \tag{3.69}$$

Now it is easy to verify (3.47). This shows that (ii) is a special case of (i).

(iii) Y **is reflexive.** By (3.59) and (3.56) we note that, for all n,

$$\gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1} \right] h_n'' \leq_C y_1 - y_{n+1} \leq_C y_1 - u_{n+1}. \tag{3.70}$$

Using (2.5), we have

$$\zeta \cdot \gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1} \right] \|h_n''\| \leq \|y_1 - u_{n+1}\| \leq \|y_1\| + \eta. \tag{3.71}$$

Thus $\{h_n''\}$ is bounded and so has a weak-cluster point $\bar{h}_0$ in the closed convex set H (thanks to the reflexivity assumption) We claim that

$$(\bar{a}, y_1 - \gamma d(z_1, \bar{a}) \bar{h}_0) \in epi\, F \tag{3.72}$$

(that is, (3.65) holds with h_0 replaced by $\bar{h}_0$ and so one completes the proof as that in (i)). To establish the claim, we take any $n_1 \in \mathbb{N}$ and let $H_1 := co(\{h_n''\}_{n > n_1})$. Then $cl(H_1)$ is a weak closed convex set and hence $\bar{h}_0 = w^*\text{-}\lim_{n \to +\infty} h_n'' \in cl(H_1)$. Therefore there exist $n_2 > n_1$ and $\lambda_{n_1+1}, \cdots, \lambda_{n_2} \geq 0$ such that $\sum_{i=n_1+1}^{n_2} \lambda_i = 1$ and

$$\|\bar{h}_0 - \bar{h}_1\| < 1, \tag{3.73}$$

where $\bar{h}_1 := \sum_{i=n_1+1}^{n_2} \lambda_i h_i'' \in H_1 \subset H$. By (3.60) we note that for any $n \in \overline{n_1 + 1, n_2}$,

$$\begin{aligned} y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n_1+1} \right] h_n'' &\geq_C y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n+1} \right] h_n'' \\ &\geq_C y_{n+1} \geq_C y_{n_2+1}, \end{aligned} \tag{3.74}$$

and hence

$$y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n_1+1} \right] \bar{h}_1 \in y_{n_2+1} + C \subset F(z_{n_2+1}) + C. \tag{3.75}$$

Inductively, we construct sequences $\{n_k\}_{k \in \mathbb{N}}$ and $\{\bar{h}_k\}_{k \in \mathbb{N}}$ such that

$$1 \leq n_1 < n_2 < \cdots, \tag{3.76}$$

$$\|\bar{h}_0 - \bar{h}_k\| < \frac{1}{k}, \qquad \forall k \in \mathbb{N}, \tag{3.77}$$

and

$$y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n_k + 1} \right] \bar{h}_k \in F(z_{n_{(k+1)}+1}) + C, \qquad \forall k \in \mathbb{N}. \tag{3.78}$$

Therefore $\left(z_{n_{(k+1)}+1}, y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n_k+1} \right] \bar{h}_k \right) \in epi\, F$ for all $k \in \mathbb{N}$ and $\left\{ \left(z_{n_{(k+1)}+1}, y_1 - \gamma \left[d(z_1, \bar{a}) - \frac{1}{n_k+1} \right] \bar{h}_k \right) \right\}_{k \in \mathbb{N}}$ converges to $(\bar{a}, y_1 - \gamma d(z_1, \bar{a}) \bar{h}_0)$. Thus (3.72) holds since $epi\, F$ is closed.

□

Remark 3.3.

(a) *The idea of using partial order in the Ekeland's Variational Principle related results has been used by many authors including Ekeland [14] himself, Bishop-Phelps [6], Brondsted-Rockafellar [8], Dancs-Hegedus-Medvegyev [11], Turinici [35], and Hamel-Tammer [18].*

(b) *Most of the extensions (see [3, Theorem 1],[4, Theorem 3.4], [9, Chpater 4] [16, Theorem 3.1 & 3.2], [18, Theorem 4.1 & 4.2], [38, Theorem 4.1 & 4.2]) and references therein) of the Ekeland principle take the following form: Under some suitable conditions, for any* $\gamma > 0$, $\xi \in C \backslash \{0\}$, $(x_0, y_0) \in graphF \subset M \times Y$, *there exists* $(\bar{x}, \bar{y}) \in graphF$ *such that*

$$\begin{aligned} &y_0 \in \bar{y} + \gamma d(x_0, \bar{x})\xi + C, \qquad \bar{y} \in Min(F(\bar{x}), \leq_C), \\ &\bar{y} \notin y + \gamma d(\bar{x}, x)\xi + C, \quad \forall (x, y) \in graphF, (x, y) \neq (\bar{x}, \bar{y}). \end{aligned} \tag{3.79}$$

A novelty of our approach here is to replace the singleton $\{\xi\}$ *but allowing* ξ *in* (3.79) *to be selected from a set H satisfying (A1) and (A2). Thus, instead of approaches of earlier authors asserting relations between elements of the values of the set-valued function F, our extension (reported in Theorem 3.4 and Theorem 3.5) of the Ekeland principle are directly expressed in terms of values (as entities) of F. The idea of replacing a singleton by a set or a suitable set-valued function has very recently been used by Bednarczuk-Zagrodny [5] and Gutiérrez-Jiménez-Novo [15] in their extended Ekeland's variational principles for vector-valued (single-valued) functions. Our discussion on the issue of error bounds in the next section will further shed light on using a set H instead of a singleton in our extensions of the Ekeland's variational principle.*

The notion of considering relations between values of F has also been used by Kuroiwa [21, 22] when he studied set-valued optimization problems and their dual problems.

(c) *The assumption (A2) plays an important role in Theorem 3.5 especially for* (3.51) *and also it ensures that* $\preceq_{(F,\gamma,H)}$ *and* $\precsim_{(F,\gamma,H)}$ *defined in Definition 3.1 are anti-symmetric.*

4. Error Bounds of Systems. In this section we consider a Banach space X and study the inequality system (1.3) defined by proper lsc functions from X to $(-\infty, +\infty]$. To avoid the triviality, we always assume that

$$D := \cap_{i=1}^{n} dom\, f_i \neq \emptyset. \tag{4.1}$$

Let S be the solution set (defined in (1.5)). For each $x \in X$, Let $I_>(x)$ denote the set of "infeasibility indices" for x while $I_\geq(x)$ denotes that of "boundary indices"; they are respectively defined by

$$\begin{aligned} I_>(x) &:= \{i \in \overline{1,n} : f_i(x) > 0\}, \\ I_\geq(x) &:= \{i \in \overline{1,n} : f_i(x) \geq 0\}. \end{aligned} \tag{4.2}$$

Let $\gamma > 0$, and we say that $u \in X \backslash S$ has the γ-descent property if there exists $\hat{u} \in X \backslash \{u\}$ satisfying the following properties

$$I_>(\hat{u}) \subset I_>(u), \tag{4.3}$$

$$f_i(\hat{u}) \leq f_i(u), \qquad \forall i \in I_>(u), \tag{4.4}$$

$$\gamma d(u, \hat{u}) \leq \sum_{i \in I_>(u)} \left(f_i(u)_+ - f_i(\hat{u})_+\right). \tag{4.5}$$

By (4.3) and (4.4) it is clear that

$$f_i(\hat{u})_+ \leq f_i(u)_+, \qquad \forall i \in \overline{1,n} \tag{4.6}$$

(both sides are zero if $i \notin I_>(u)$). This together with (4.5) implies that

$$\gamma d(u, \hat{u}) \leq \sum_{i=1}^{n} \left(f_i(u)_+ - f_i(\hat{u})_+\right). \tag{4.7}$$

EXAMPLE 4.1. *Let f be be a proper function from X to $(-\infty, +\infty]$. For $x \in X$ with $f(x) \in \mathbb{R}$ and $v \in X$, recall that (cf. [28]) upper Dini-directional derivative of f at x in direction v is defined by*

$$\bar{d}^+ f(x)(v) := \limsup_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}. \tag{4.8}$$

Let $\gamma > 0$ and $u \in X \backslash S$. If there exists $v \in X$ with $\|v\| = 1$ such that

$$\bar{d}^+ f_i(u)(v) \text{ exists and finite for each } i \in \overline{1,n}, \tag{4.9}$$

$$\bar{d}^+ f_i(u)(v) < 0, \qquad \forall i \in I_\geq(u), \tag{4.10}$$

$$\sum_{i \in I_>(u)} \bar{d}^+ f_i(u)(v) \leq -\gamma. \tag{4.11}$$

Then u has the γ'-descent property for any $\gamma' \in (0, \gamma)$. To see this, let $\gamma' \in (0, \gamma)$ be fixed. By (4.11) and (4.10) (applied to the indices i in $I_>(u)$), there exists a series of positive numbers $\{\gamma_i' : i \in I_>(u)\}$ such that

$$\bar{d}^+ f_i(u)(v) < -\gamma_i', \qquad \forall i \in I_>(u), \tag{4.12}$$

and

$$\sum_{i \in I_>(x)} \gamma_i' = \gamma'. \tag{4.13}$$

For each $i \in \overline{1,n}$, we select $t_i > 0$ in the following way:

(a) If $f_i(u) > 0$, then by (4.12), there exists $t_i > 0$ such that

$$0 < f_i(u + tv) < f_i(u) - \gamma_i' t, \qquad \forall t \in (0, t_i]. \tag{4.14}$$

(b) If $f_i(u) = 0$ then, by (4.10), there exists $t_i > 0$ such that

$$f_i(u + tv) < 0, \qquad \forall t \in (0, t_i]. \tag{4.15}$$

(c) If $f_i(u) < 0$ then, by (4.9), there exists $t_i > 0$ such that

$$f_i(u + tv) < 0, \qquad \forall t \in (0, t_i]. \tag{4.16}$$

Having specified $t_i > 0$ for all $i \in \overline{1, n}$, let $\hat{t} := \min\{t_i : i \in \overline{1, n}\}$ and $\hat{u} := u + \hat{t}v$. We note that the following equivalence holds for all i:

$$f_i(\hat{u}) > 0 \Longleftrightarrow f_i(u) > 0,$$

that is $I_>(\hat{u}) = I_>(u)$((4.3) holds). Further, by (4.13) and (4.14), we have (4.4) and

$$\gamma' d(u, \hat{u}) \leq \sum_{i \in I_>(u)} (f_i(u)_+ - f_i(\hat{u})_+). \tag{4.17}$$

Therefore u has the γ'-descent property.

The following result was established (based on a result of Hamel [17, Theorem 2(ii)]) in [28] for the special case when $n = 1$ and $W = \emptyset$.

THEOREM 4.1. *Let X, $f_1, f_2, \cdots, f_n$, D and S be as at the beginning of this section. Suppose that there exist positive constants τ_1, γ and a subset W of X satisfying the following two conditions:*

(i)

$$d(w, S) \leq \tau_1 \sum_{i=1}^{n} f_i(w)_+, \qquad \forall w \in W. \tag{4.18}$$

(ii) Each $u \in X \backslash (W \cup S)$ has the γ-descend property (with the corresponding $\hat{u} \neq u$).

Then the inequality system (1.3) has an error bound $\tau := \max\{\gamma^{-1}, \tau_1\}$.

Proof. Let x_0 be an arbitrary element of X. We have to show that the following inequality

$$d(x_0, S) \leq \tau \sum_{i=1}^{n} f_i(x_0)_+. \tag{4.19}$$

We suppose without loss of generality that $x_0 \in D \backslash (W \cup S)$. We shall apply Theorem 3.5 with the following data: $Y = \mathbb{R}^n$ with the partial order defined by $C := \mathbb{R}^n_+$ and the l_1-norm. Clearly, if $y \in \mathbb{R}^n_+ \backslash \{0\}$ and $\lambda \geq 0$ then

$$\lambda \leq \|y\|_1 \Longleftrightarrow \lambda \frac{y}{\|y\|_1} \leq_{\mathbb{R}^n_+} y. \tag{4.20}$$

Let $H := \mathbb{R}^n_+ \cap \{y : \|y\|_1 = 1\}$. Thus

$$d(0, H) = 1, \quad \overline{cone} H = \mathbb{R}^n_+ \quad \text{and} \quad \measuredangle(\overline{cone} H, -\mathbb{R}^n_+) = 1. \tag{4.21}$$

Let $F : D \to Y$ be defined by $F(x) = (f_1(x)_+, f_2(x)_+, \cdots, f_n(x)_+)$ for $x \in D$. Thus $epi\, F$ is closed (thanks to the assumption that each f_i is lsc). Together with (4.21), we see that (A1)-(A4) are satisfied by the data. Hence Proposition 3.2 and Theorem

3.5 are applicable with $\preceq:=\preceq_{(F,\gamma,H)}$. In particular, there exists $u \in D$ satisfying $u \preceq x_0$ and $z \not\preceq u$ for all $z \in D\backslash\{u\}$ (that is u is a minimum element of D). Thus, by definitions, there exists $h_1 \in H$ such that

$$\gamma d(x_0, u)h_1 \leq_{\mathbb{R}^n_+} F(x_0) - F(u), \tag{4.22}$$

and so

$$\gamma d(x_0, u) = \|\gamma d(x_0, u)h_1\|_1 \leq \|F(x_0) - F(u)\|_1. \tag{4.23}$$

We claim that $u \in W \cup S$. If not, then, by assumption (ii), u has the γ-descent property with the corresponding $\hat{u} \neq u$: (4.3)-(4.7) hold. Clearly, (4.7) can be rewritten as $\gamma d(u, \hat{u}) \leq \|F(u) - F(\hat{u})\|_1$, and it follows from (4.20) that $\gamma d(u, \hat{u})h_2 \leq F(u) - F(\hat{u})$, where $h_2 := \frac{F(u)-F(\hat{u})}{\|F(u)-F(\hat{u})\|_1}$. This implies that $\hat{u} \preceq u$, contradicting the minimality of u. This shows that $u \in W \cup S$. If $u \in S$ then $F(u) = 0$ and (4.23) entails that

$$\gamma d(x_0, S) \leq \|F(x_0)\|_1 = \sum_{i=1}^{n} f_i(x_0)_+. \tag{4.24}$$

and so (4.19) holds in this case. It remains to consider the case when $u \in W$. Then, by (4.18), one has $\tau_1^{-1} d(u, S) \leq \|F(u)\|_1$ and it follows from (4.20) that, for $h_3 = \frac{F(u)}{\|F(u)\|_1}$,

$$\tau_1^{-1} d(u, S)h_3 \leq_{\mathbb{R}^n_+} F(u). \tag{4.25}$$

This and (4.22) imply that

$$\min\{\gamma, \tau_1^{-1}\}[d(x_0, u)h_1 + d(u, S)h_3] \leq_{\mathbb{R}^n_+} F(x_0). \tag{4.26}$$

and so $\min\{\gamma, \tau_1^{-1}\}[d(x_0, u) + d(u, S)] \leq \|F(x_0)\|_1$. Thus (4.19) is seen to be true. □

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