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Extended Newton methods for conic inequalities: Approximate solutions and the extended Smale α-theory

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ABSTRACT

Using the convex process theory and the majorizing technique, we study the convergence issues of the iterative sequences generated by the extended Newton method for solving the conic inequality system $F \geq_C 0$ defined by a cone C and a Fréchet differentiable function F satisfying the (extended) weak γ-condition. Convergence criterion of the extended Newton method is presented. As an application, we establish, when F is analytic, an extended α-theory similar to Smale's α-theory for nonlinear analytic equations.

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1. Introduction

We study the convergence issue of the extended Newton method (Algorithm $\mathbf{A}(x_0)$ below, with a suitable starting point x_0) for the problem

$$F(x) \geq_C 0, \tag{1.1}$$

or, as a special kind,

$$F(x) = 0, \tag{1.2}$$

where $F : \Omega \subseteq X \to Y$ is a continuously differentiable function, X, Y are Banach spaces, Ω is an open subset, and $\geq_C$ is a partial order (or preorder) defined by a closed convex cone C in Y:

$$y_1 \geq_C y_2 \Longleftrightarrow y_1 - y_2 \in C. \tag{1.3}$$

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Newton's method is probably the most important method for solving problem (1.2), which proceeds as follows: for any starting point $x_0 \in \Omega$, construct iteratively a sequence such that

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad \text{for each } n = 0, 1, \ldots. \tag{1.4}$$

An important result on Newton's method is the well-known Kantorovich theorem (cf. [13]), which provides a simple and clear criterion ensuring the quadratic convergence of Newton's method under some mild condition such as that the Fréchet derivative $F'(x_0)$ of F at starting point x_0 is nonsingular and the second Fréchet derivative of F is bounded on an appropriate metric ball of the starting point x_0 contained in Ω. Another important result on Newton's method is the famous Smale α-theory (with analytic F), presented in the report [29] (see also [28,30]), studying the notion of approximate zeros (in section 4 we briefly recall this important notion), and establishing rules to decide if a starting point x_0 is an approximate zero. One assumes that $F'(x_0)$ is nonsingular and defines

$$\alpha(F, x) := \beta(F, x)\gamma(F, x)$$

with

$$\gamma(F, x) := \sup_{k\geq 2} \| \, F'(x)^{-1}\frac{F^{(k)}(x)}{k!} \, \|^{\frac{1}{k-1}} \qquad \text{and} \qquad \beta(F, x) := \|F'(x)^{-1}F(x)\|.$$

Then the rule is set to depend on information $\alpha(F, \cdot)$ at this starting point x_0: if $x_0 \in X$ is such that

$$\alpha(F, x_0) < \alpha_0, \tag{1.5}$$

with $\alpha_0 := 0.130716944\cdots$ being the unique root of the equation $2u = (1 - 4u + 2u^2)^2$ in $(0, 1 - \frac{\sqrt{2}}{2})$, then Newton's method with the starting point x_0 is well defined (namely (1.4) generates a unique sequence $\{x_n\}$) and satisfies that

$$\|x_{n+1} - x_n\| \leq \left(\frac{1}{2}\right)^{2^n-1} \|x_1 - x_0\| \quad \text{for any } n \geq 0. \tag{1.6}$$

Since then, this line of research has been extensively studied resulting many significant improvements and extensions in several directions; see for example [4,6–9,26,32,34] and references therein. In particular, Smale's result was sharpened in [32] in such a way that the criterion (1.5) is replaced by the following weaker condition:

$$\alpha(F, x_0) \leq \frac{13 - 3\sqrt{17}}{4} = 0.157671\cdots; \tag{1.7}$$

while (1.6) by the following stronger one:

$$\|x_{n+1} - x_n\| \leq \left(\frac{1}{2}\right)^{2^n-1} \|x_n - x_{n-1}\| \quad \text{for any } n \geq 0, \tag{1.8}$$

that is x_0 is an approximate zero of F.

When X, Y are Euclidean spaces, these results were extended by Shub and Smale [26] to cover the case of underdetermined analytic system defined by F such that, at the starting point x_0,

$$F'(x_0) \text{ is surjective} \tag{1.9}$$

(but $F'(x_0)$ is not necessarily injective, and so they used the notion of the corresponding Moore–Penrose inverses to replace that of the inverses in the definitions of $\gamma(F,\cdot)$ and $\beta(F,\cdot)$).

Another direction of extending Smale's theory concerns with the general case when the analytic assumption is dropped. This was done in [33] (see also [31]) by making use of the weak γ-condition for nonlinear C^2-operators F between Banach spaces. The notion of the weak γ-condition was also used in [12,19,17] to extend and improve the corresponding results in [26,7,5], respectively. More extensions of this idea to the Gauss–Newton method for convex composite optimization are referred to [10] and [15].

Concerning the issue of solving (1.1), Robinson proposed in [22] the following Algorithm $\mathbf{A}(x_0)$ (which is called the extended Newton method) with starting point $x_0 \in \Omega$. For $x \in \Omega$, we define

$$\mathfrak{D}(x) := \{d \in X :\ F(x) + F'(x)d \in C\}. \tag{1.10}$$

Moreover, for any subset D of a normed space, $\|D\|$ denotes the distance from D to the origin, namely,

$$\|D\| := \mathrm{d}(0, D) = \inf\{\|a\| :\ a \in D\} \le +\infty \tag{1.11}$$

(so $\|D\| < +\infty$ if and only if D is nonempty).

Algorithm $\mathbf{A}(x_0)$. For $k = 0, 1, \cdots$, having x_k, determine x_{k+1} as follows:
Let $d_k \in \mathfrak{D}(x_k)$ exist such that $\|d_k\| = \|\mathfrak{D}(x_k)\|$. Then pick such d_k and set $x_{k+1} = x_k + d_k$.

If such d_k does exist for all k, then x_0 is called an implementable starting point for Algorithm $\mathbf{A}(x_0)$. In this paper we present a point estimate theory (which maybe called an extended Smale α-theory) in which we provide some sufficient conditions ensuring that a starting point x_0 is an implementable one for Algorithm $\mathbf{A}(x_0)$ and any sequence $\{x_n\}$ so generated converges to a solution x^* for (1.1) giving estimates how rapid of the convergence. Specializing to problem (1.2) (that is $C := \{0\}$), results reported here recapture the corresponding results of Shub and Smale [26] but in a broad context, namely assumption (1.9) is dropped and X and Y are not necessarily Euclidean spaces in our consideration.

Our main results are the following Theorems 1.1 and 1.2, applicable to C^2-functions and analytic functions respectively. For each $x \in \Omega$, let $T_x : X \to 2^Y$ be the convex process defined by

$$T_x d := F'(x)d - C \quad \text{for each } d \in X, \tag{1.12}$$

so $T_x^{-1} : Y \to 2^X$ is the convex process defined by

$$T_x^{-1} y := \{d \in X :\ F'(x)d \in y + C\} \quad \text{for each } y \in Y \tag{1.13}$$

and

$$T_x^{-1}(-F(x)) = \mathfrak{D}(x). \tag{1.14}$$

Henceforth, we assume that X is reflexive and $x_0 \in \Omega$ is such that

$$F(x_0) + F'(x_0)d \in C \quad \text{for some } d \in X. \tag{1.15}$$

Let

$$\xi := \|T_{x_0}^{-1}(-F(x_0))\|, \tag{1.16}$$

the distance to the origin from the set $T_{x_0}^{-1}(-F(x_0))$ and we fix $\gamma \ge 0$. Let r^* be defined by

$$r^* := \begin{cases} \frac{1+\gamma\xi-\sqrt{(1+\gamma\xi)^2-8\gamma\xi}}{4\gamma} & \text{if } \gamma > 0, \\ \xi & \text{if } \gamma = 0. \end{cases} \tag{1.17}$$

Moreover, if F is C^k at x, we write

$$\|T_{x_0}^{-1}F^{(k)}(x)\| := \sup\{\|T_{x_0}^{-1}(F^{(k)}(x)(z_1, z_2, \cdots, z_k))\| : \{z_i\}_{i=1}^k \subset \overline{\mathbf{B}(0,1)}\}, \tag{1.18}$$

where, as usual, $\overline{\mathbf{B}(x,r)}$ denotes the closure of the open ball $\mathbf{B}(x,r)$ with center x and radius r. Note that

$$\mathrm{R}(F^{(k)}(x)) \subseteq \mathrm{R}(T_{x_0}) \qquad \text{whenever } \|T_{x_0}^{-1}F^{(k)}(x)\| < +\infty \tag{1.19}$$

(see (2.1) and (3.3) in next sections for the definition of the ranges $\mathrm{R}(T_{x_0})$) and $\mathrm{R}(F^{(k)}(x))$.

Theorem 1.1. *Let $\Omega \supseteq \overline{\mathbf{B}(x_0, r^*)}$, and suppose that F is C^2 on $\overline{\mathbf{B}(x_0, r^*)}$. Suppose that*

$$\xi \le \frac{3-2\sqrt{2}}{\gamma}, \tag{1.20}$$

and that (T_{x_0}, F) satisfies the weak γ-condition at x_0 on $\mathbf{B}(x_0, r^*)$:

$$\|T_{x_0}^{-1}F''(x)\| \le \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3} \quad \text{for each } x \in \mathbf{B}(x_0, r^*). \tag{1.21}$$

Then x_0 is a implementable starting point for Algorithm $\mathbf{A}(x_0)$, and any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$ is contained in $\overline{\mathbf{B}(x_0, r^*)}$ and converges to a solution x^* of (1.1) satisfying

$$\|x_{n+1} - x_n\| \le q^{2^n-1}\|x_n - x_{n-1}\| \quad \text{for each } n = 1, 2, \ldots \tag{1.22}$$

and

$$\|x_n - x^*\| \le q^{2^n-1}r^* \quad \text{for each } n = 0, 1, 2, \ldots, \tag{1.23}$$

where q is a constant such that $0 \le q < 1$; in fact, q can be given by

$$q := \frac{1-\gamma\xi-\sqrt{(1+\gamma\xi)^2-8\gamma\xi}}{1-\gamma\xi+\sqrt{(1+\gamma\xi)^2-8\gamma\xi}}, \tag{1.24}$$

provided that $0 < \gamma < +\infty$.

Theorem 1.2. *Let $\Omega \supseteq \overline{\mathbf{B}(x_0, r^*)}$. Suppose that F is analytic on Ω,*

$$\xi \le \frac{13-3\sqrt{17}}{4\gamma}, \tag{1.25}$$

and

$$\gamma = \sup_{k\ge 2} \left\| \frac{T_{x_0}^{-1}F^{(k)}(x_0)}{k!} \right\|^{\frac{1}{k-1}} < +\infty. \tag{1.26}$$

Then, assumptions (1.20) and (1.21) hold, and x_0 is an approximate solution of (1.1), that is, x_0 is an implementable starting point for Algorithm $\mathbf{A}(x_0)$ with any generated sequence $\{x_n\}$ satisfying (1.8). Moreover, any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$ is contained in $\overline{\mathbf{B}(x_0, r^*)}$.

To extend the Smale α-theory in [26] for underdetermined equation systems to the case when X and Y are not necessarily finite-dimensional and (1.9) is not necessarily satisfied, we introduce the notion of generalized inverses as follows. Let X be a Hilbert space and $A : X \to Y$ be a bounded linear operator such that $\mathrm{R}(A)$ is complemented in Y (in the sense that there exists a bounded linear projection operator $Q : X \to \mathrm{R}(A)$), where $\mathrm{R}(T)$ denotes the image of an operator T. Then, by [21], there exists a bounded linear operator (called a generalized inverse of A and denoted by A^+) from Y into X such that

$$A\,A^+\,A = A, \quad A^+\,A\,A^+ = A^+ \quad \text{and} \quad A^+\,A = \mathbf{I} - \Pi_{\ker A}, \tag{1.27}$$

where $\Pi_{\ker A}$ is the orthogonal projection on $\ker A$ and $\mathbf{I}$ is the identify operator on X.

Applying Theorem 1.2 to the following extended Newton method for solving problem (1.2):

$$x_{n+1} := x_n - F'(x_n)^+ F(x_n) \quad \text{for any } n = 0, 1, \ldots, \tag{1.28}$$

we have the following corollary.

Corollary 1.1. *Let $\Omega \supseteq \overline{\mathbf{B}(x_0, r^*)}$. Suppose that X is a Hilbert space and F is analytic on Ω such that $\mathrm{R}(F'(x_0))$ is complemented in Y. Let*

$$\xi := \|F'(x_0)^+ F(x_0)\| \quad \textit{and} \quad \gamma := \sup_{k\geq 2} \left\| \frac{F'(x_0)^+ F^{(k)}(x_0)}{k!} \right\|^{\frac{1}{k-1}} \tag{1.29}$$

be such that (1.25) holds. Suppose further that

$$F(x_0) \in \mathrm{R}(F'(x_0)) \quad \textit{and} \quad \mathrm{R}(F^{(k)}(x_0)) \subseteq \mathrm{R}(F'(x_0)) \quad \textit{for any } k \in \mathbb{N} \tag{1.30}$$

(of course, they are satisfied trivially if $F'(x_0)$ is surjective). Then, x_0 is an approximate zero of F, that is, the extended Newton method (1.28) is well-defined and the generated sequence $\{x_n\}$ satisfies (1.8).

Another motivation of the present paper stems from [5] for finite inequality systems and [17] for the conic inequality system (1.1), where the Smale α-theory was used to establish the existence and the error bound results for the solution of (1.1). As an applications of the extended Smale α-theory in the present paper, we will study in our next paper [18] the issue of the error bounds for not only the conic inequality system (1.1) but also perturbed systems arising from (1.1), and will establish some quantitative results on the error bounds, the calmness property and the Lipschitz-like continuity for the solution maps of these systems. In particular, some corresponding results in [2,3,5,11,17,20] will be extended and improved.

The paper is organized as follows. In section 2, we list some basic concepts and known facts needed in the sequel. In section 3 we extend the notion of the γ-condition introduced in [33] for operators to the case of convex processes, together with some related results, and then provide the proof for Theorem 1.1. The proofs for Theorem 1.2 and Corollary 1.1 are given in section 4.

2. Preliminaries

We always assume that X, Y, Z are Banach spaces. Fix $x \in X$. Let $\mathbf{B}(x, r)$ stand for the open ball in X or Y with center x and radius r; while the corresponding closed ball is denoted by $\overline{\mathbf{B}(x, r)}$. As usual, the space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$, endowed with the operator norm.

The concept of convex process (which was introduced by Rockafeller [25,24] and extensively used by Robinson [22,23] for problem (1.1)) plays a key role in the study of this paper.

Definition 2.1. A set-valued map $T : X \to 2^Y$ is called a convex process from X to Y if it satisfies

(a) $T(x + y) \supseteq Tx + Ty$ for all $x, y \in X$;
(b) $T(\lambda x) = \lambda Tx$ for all $\lambda > 0$, $x \in X$;
(c) $0 \in T0$.

Thus $T : X \to 2^Y$ is a convex process if and only if its graph $\mathrm{Gr}(T) := \{(x, y) \in X \times Y : y \in Tx\}$ is a convex cone in $X \times Y$. By definition, a convex process $T : X \to 2^Y$ is closed if its graph $\mathrm{Gr}(T)$ is closed. As usual, the domain, range and inverse of a convex process T are respectively denoted by $\mathrm{D}(T)$, $\mathrm{R}(T)$ and T^{-1}; i.e.,

$$\begin{aligned}\mathrm{D}(T) &:= \{x \in X : Tx \neq \emptyset\},\\ \mathrm{R}(T) &:= \bigcup\{Tx : x \in \mathrm{D}(T)\}\end{aligned} \tag{2.1}$$

and

$$T^{-1}y := \{x \in X : y \in Tx\} \quad \text{for each } y \in Y.$$

Obviously T^{-1} is a convex process from Y to X. Recall that, for a subset D of a normed space, $\|D\|$ is its distance to the origin. We also make the convention that $D + \emptyset = \emptyset$ for each set D. By definition, the following inequality holds for a convex process T:

$$\|T(x + y)\| \leq \|Tx\| + \|Ty\| \quad \text{for any } x, y \in \mathrm{D}(T). \tag{2.2}$$

Definition 2.2. Suppose that T is a convex process. The norm of T is defined by

$$\|T\| := \sup\{\|Tx\| : x \in \mathrm{D}(T), \ \ \|x\| \leq 1\}.$$

If $\|T\| < +\infty$, we say that the convex process T is normed.

Proposition 2.1. *Suppose that Y is reflexive and that $T : Y \to 2^Z$ is a closed convex process. If $\|T^{-1}\| < \infty$, then $\mathrm{R}(T)$ is closed.*

Proof. Let $\{z_n\} \subseteq \mathrm{R}(T)$ be such that $z_n \to z_0$. Then $\{z_n\} \subseteq \mathrm{D}(T^{-1})$. Without loss of generality, we may assume that $\|z_n\| = \|z_0\| = 1$. Since $\|T^{-1}\| < \infty$, we can take $\{y_n\} \subseteq Y$ such that $\{y_n\}$ is bounded and $y_n \in T^{-1}z_n$ for each n. Since Y is reflexive, by the Eberlein–Smulian Theorem in Functional Analysis (cf. [35, p. 141]), we may assume that, without loss of generality (using a subsequence if necessary), $\{y_n\}$ converges weakly to one point in Y, say y_0. Consequently, it follows from the Mazur Theorem in Functional Analysis (cf. [35, p. 120]) that there exists a sequence $\{\tilde{y}_k\}$ with the convex expression

$$\tilde{y}_k := \sum_{i=1}^{n_k} \alpha_i^k y_{k_i} \quad \text{for each } k = 1, 2, \cdots,$$

where $\{\alpha_i^k\} \subseteq [0, 1]$ satisfies $\sum_{i=1}^{n_k} \alpha_i^k = 1$ for each k, such that $\{\tilde{y}_k\}$ converges in norm to y_0 and the corresponding sequence $\{\tilde{z}_k\}$ generated by the convex combinations of $\{z_k\}$ converges to z_0, that is,

$$\tilde{z}_k := \sum_{i=1}^{n_k} \alpha_i^k z_{k_i} \to z_0.$$

Note by definition that

$$\tilde{z}_k \in \sum_{i=1}^{n_k} \alpha_i^k T y_{k_i} \subseteq T\tilde{y}_k.$$

It follows from the closedness assumption of T that $z_0 \in Ty_0$; hence $z_0 \in \mathrm{R}(T)$ and the proof is complete. □

Let $T,\ S : X \to 2^Y$ and $Q : Y \to 2^Z$ be convex processes. Recall that $T \subseteq S$ means that $\mathrm{Gr}(T) \subseteq \mathrm{Gr}(S)$, that is, $Tx \subseteq Sx$ for each $x \in \mathrm{D}(T)$. By definition, one can verify easily that $\|T\| \geq \|S\|$ if $T \subseteq S$ and $\mathrm{D}(T) = \mathrm{D}(S)$. Moreover, $T \subseteq S$ if and only if $T^{-1} \subseteq S^{-1}$. The sum $T + S$, composite $Q\,S$ and multiple λT are processes defined respectively by

$$(T + S)(x) := Tx + Sx \quad \text{for each } x \in X,$$
$$Q\,S(x) = Q(S(x)) := \bigcup_{y \in S(x)} Q(y) \quad \text{for each } x \in X$$

and

$$(\lambda T)(x) := \lambda(Tx) \quad \text{for each } x \in X.$$

It is well known (and easy to verify) that $T + S$, $Q\,S$, λT are also convex processes and the following assertions hold:

$$\|T + S\| \leq \|T\| + \|S\|, \qquad \|Q\,S\| \leq \|Q\|\,\|S\| \qquad \text{and} \qquad \|\lambda T\| = |\lambda|\|T\|. \tag{2.3}$$

We also require two propositions below: the first one is known in [23] while the second is a direct consequence of the first one and [22, Theorem 5].

Proposition 2.2. *Let $T : X \to 2^Y$ be a closed convex process. Then we have the following assertions:*

(i) *If $\mathrm{D}(T) = X$, then T is normed.*
(ii) *If $\mathrm{R}(T) = Y$, then T^{-1} is normed. Consequently, T^{-1} is normed if $\mathrm{R}(T)$ is a closed linear subspace.*

Proposition 2.3. *Let $S_1,\ S_2 : X \to 2^Y$ be closed convex processes with $\mathrm{D}(S_1) = \mathrm{D}(S_2) = X$ and $\mathrm{R}(S_1) = Y$. Suppose that $\|S_1^{-1}\|\|S_2\| < 1$ and that $(S_1 + S_2)(x)$ is closed for each $x \in X$. Then $\mathrm{R}(S_1 + S_2) = Y$ and $\|(S_1 + S_2)^{-1}\| \leq \frac{\|S_1^{-1}\|}{1-\|S_1^{-1}\|\|S_2\|}$.*

The following lemma is a two-dimensional extension of its one-dimensional version given in [16] and we shall omit its proof which is similar to that in [16, Lemma 2.1].

Lemma 2.1. *Let $g : [0, 1] \times [0, 1] \to \mathbb{R}$ and $G : [0, 1] \times [0, 1] \to Y$ be continuous. Let Z be a reflexive Banach space. Suppose that $T : Y \to 2^Z$ is a closed convex process such that*

$$\|TG(t, s)\| \leq g(t, s) \quad \textit{for each } (t, s) \in [0, 1] \times [0, 1]. \tag{2.4}$$

Then $T \iint\limits_{[0,1]^2} G(t, s)\,\mathrm{d}t\mathrm{d}s \neq \emptyset$ and

$$\left\| T \iint\limits_{[0,1]^2} G(t, s)\,\mathrm{d}t\mathrm{d}s \right\| \leq \iint\limits_{[0,1]^2} g(t, s)\,\mathrm{d}t\mathrm{d}s. \tag{2.5}$$

Let ψ and η denote the functions defined by

$$\psi(\tau) := \frac{\tau-1}{\tau(2\tau-1)} \quad \text{and} \quad \eta(t) := \frac{2}{1+t+\sqrt{(1+t)^2-8t}} \quad \text{for any } \tau \in [1, \frac{2+\sqrt{2}}{2}],\ t \in [0, 3-2\sqrt{2}]. \quad (2.6)$$

By differential calculus, one can verify that they are (strictly) increasing functions such that ψ maps $[1, \frac{2+\sqrt{2}}{2}]$ onto $[0, 3-2\sqrt{2}]$ and $\psi^{-1} = \eta$.

Assume, for the remainder of this subsection, that (γ, ξ) is a fixed pair of constants such that

$$\gamma \geq 0, \quad \xi \geq 0 \quad \text{and} \quad \gamma\xi \leq 3-2\sqrt{2}. \quad (2.7)$$

We also adopt the following convention:

$$0^0 := 1 \quad \text{and} \quad \frac{a}{0} := +\infty \quad \text{for any } a > 0. \quad (2.8)$$

Define the function $\phi_{\gamma,\xi}$ (to be abbreviated to ϕ for short) by

$$\phi(t) := \xi - t + \frac{\gamma t^2}{1-\gamma t} \quad \text{for each } t \in [0, \frac{1}{\gamma}). \quad (2.9)$$

Then, one has

$$\phi'(t) = -2 + \frac{1}{(1-\gamma t)^2} \quad \text{and} \quad \phi''(t) = \frac{2\gamma}{(1-\gamma t)^3} \quad \text{for each } t \in [0, \frac{1}{\gamma}) \quad (2.10)$$

(in particular, ϕ' and ϕ'' do not depend on ξ). Assuming $\gamma \neq 0$, let

$$r_0 := \frac{2-\sqrt{2}}{2\gamma} \quad \text{and} \quad r^* := \frac{1+\gamma\xi - \sqrt{(1+\gamma\xi)^2 - 8\gamma\xi}}{4\gamma}, \quad (2.11)$$

that is, r^* is the smaller zero of ϕ in $[0, +\infty)$. Thus, by (2.7) and (2.11), we have that

$$r^* \leq \frac{1+\gamma\xi}{4\gamma} \leq \frac{1+(3-2\sqrt{2})}{4\gamma} = \frac{2-\sqrt{2}}{2\gamma} = r_0, \quad (2.12)$$

and

$$r^* < r_0 \quad \text{if} \quad \gamma\xi < 3-2\sqrt{2} \quad (2.13)$$

(so $r^* < \frac{1}{\gamma}$ as $\frac{2-\sqrt{2}}{2} < 1$). Note that r_0 is the unique zero of ϕ' in $[0, \frac{1}{\gamma})$ and that ϕ is decreasing in $[0, r_0)$ while increasing in $(r_0, \frac{1}{\gamma})$. Let $\{t_n\}$ be the sequence generated by Newton's method for ϕ with starting point $t_0 = 0$:

$$t_{n+1} := t_n - \phi'(t_n)^{-1}\phi(t_n) \quad \text{for each } n = 0, 1, \ldots. \quad (2.14)$$

In particular,

$$t_1 = \xi. \quad (2.15)$$

Let

$$q := \frac{1 - \gamma\xi - \sqrt{(1+\gamma\xi)^2 - 8\gamma\xi}}{1 - \gamma\xi + \sqrt{(1+\gamma\xi)^2 - 8\gamma\xi}} \tag{2.16}$$

and

$$p := \frac{1 + \gamma\xi - \sqrt{(1+\gamma\xi)^2 - 8\gamma\xi}}{1 + \gamma\xi + \sqrt{(1+\gamma\xi)^2 - 8\gamma\xi}}. \tag{2.17}$$

Noting (2.7), it is elementary to show that

$$q \le \frac{1}{2} \Longleftrightarrow \xi \le \frac{13 - 3\sqrt{17}}{4\gamma}. \tag{2.18}$$

Furthermore, we have the following lemma, which is known in [31].

Lemma 2.2. *Under assumption (2.7), the following assertions hold:*

(i) *The sequence $\{t_n\}$ is increasing and has the closed form:*

$$r^* - t_n = \frac{1-p}{1 - q^{2^n-1}p} q^{2^n-1} r^* \le q^{2^n-1} r^* \quad \textit{for each } n = 0, 1, 2, \ldots. \tag{2.19}$$

(ii)

$$t_{n+1} - t_n \le q^{2^n-1}(t_n - t_{n-1}) \quad \textit{for each } n = 1, 2, \ldots. \tag{2.20}$$

Remark 2.1. Suppose that $\gamma = 0$. Then, by the usual convention,

$$r_0 = +\infty \quad \text{and} \quad r^* = \xi \left(= \lim_{\gamma \to 0+} \frac{1 + \gamma\xi - \sqrt{(1+\gamma\xi)^2 - 8\gamma\xi}}{4\gamma} \right); \tag{2.21}$$

thus (2.7) and (2.12) hold trivially. Also the function ϕ defined by (2.9) simply means that $\phi(t) = \xi - t$ for each $t \in [0, +\infty)$. Therefore $t_n = \xi$ for all $n \ge 1$ and $p = q = 0$. Consequently, (2.19) and (2.20) in Lemma 2.2 also hold trivially in this case.

3. Proof of Theorem 1.1

Before presenting the proof of Theorem 1.1, let us recall some notation and a preparatory lemma. As assumed in the introduction section, let X, Y be Banach spaces, $F : \Omega \subseteq X \to Y$ a continuously differentiable function, and C a nonempty closed convex cone in Y.

For $x \in \Omega$, we define two convex processes T_x and T_x^{-1} as in (1.12) and (1.13). Since $F'(x)$ is continuous and C is closed, it is easy to verify that T_x and T_x^{-1} are closed. Furthermore, for any $x_0 \in \Omega$, one can show (by a straightforward verification and making use of the fact that $C + C = C$), that

$$T_x^{-1} F'(x_0) T_{x_0}^{-1} \subseteq T_x^{-1} \tag{3.1}$$

and that

$$\mathfrak{D}(x) = T_x^{-1}(-F(x)) = T_x^{-1}(C - F(x)) \tag{3.2}$$

(recalling (1.10)). For convenience, let us introduce further notations that will be used in this paper.

Let $\mathbb{N}$ denote the set of all natural numbers and let $k \in \mathbb{N}$, and consider a k-multilinear bounded operator $\Lambda : (X)^k \to Y$. We define the norm $\|\Lambda\|$ by

$$\|\Lambda\| := \sup\{\|\Lambda(x_1, \ldots, x_k)\| : (x_1, \ldots, x_k) \in (X)^k, \|x_i\| \le 1 \text{ for each } i\};$$

also, let $\mathrm{R}(\Lambda)$ denote the image of Λ:

$$\mathrm{R}(\Lambda) := \{\Lambda(x_1, \ldots, x_k) : (x_1, \ldots, x_k) \in (X)^k\}. \tag{3.3}$$

Assume that F is C^k (kth continuously differentiable) on Ω. The kth derivative $F^{(k)}(x)$ at x is a k-multilinear operator from $(X)^k$ to Y. It follows that, for any $k-1$ points $z_1, z_2, \cdots, z_{k-1} \in X$, $T_{x_0}^{-1}(F^{(k)}(x)(z_1, z_2, \cdots, z_{k-1}))$ is a convex process from X to Y. Then

$$\|T_{x_0}^{-1}F^{(k)}(x)\| = \sup\{\|T_{x_0}^{-1}(F^{(k)}(x)(z_1, z_2, \cdots, z_{k-1}))\| : \{z_i\}_{i=1}^{k-1} \subset \overline{\mathbf{B}(0,1)}\}. \tag{3.4}$$

Note in particular that, for each $j \le k$,

$$\|T_{x_0}^{-1}F^{(k)}(x)z^j\| \le \|T_{x_0}^{-1}F^{(k)}(x)\|\|z\|^j \quad \text{for each } z \in X, \tag{3.5}$$

where and in the sequel, the z^j denotes, as usual, $(z, \cdots, z) \in (X)^j$ for each $z \in X$; moreover, if $(z_1, \cdots, z_l) \in (X)^l$, then $(z^j, z_1, \cdots, z_l)$ denotes the corresponding element in $(X)^{j+l}$. Thus, in terms of the notation $\mathrm{R}(\cdot)$, it is routine to verify that, for all x, $z \in X$, the following equivalences hold:

$$\mathrm{R}(F'(x)) \subseteq \mathrm{R}(T_z) \iff \mathrm{R}(T_x) \subseteq \mathrm{R}(T_z) \iff \mathrm{D}(T_z^{-1}F'(x)) = X. \tag{3.6}$$

Lemma 3.1 below is known in [16] (cf. the proof for [16, Proposition 2.3]).

Lemma 3.1. *Let $x_0 \in X$ and $A \in \mathcal{L}(X,Y)$ be such that $\mathrm{R}(F'(x_0) + A) \subseteq \mathrm{R}(T_{x_0})$ and $\|T_{x_0}^{-1}A\| < 1$. Define $\tilde{T}(\cdot) := T_{x_0}(\cdot) + A(\cdot)$. Then*

$$\mathrm{R}(T_{x_0}) \subseteq \mathrm{R}(\tilde{T}), \quad \mathrm{D}(\tilde{T}^{-1}F'(x_0)) = X \tag{3.7}$$

and

$$\|\tilde{T}^{-1}F'(x_0)\| \le \frac{1}{1 - \|T_{x_0}^{-1}A\|}. \tag{3.8}$$

In his study of problem (1.1), Robinson [22] required an important assumption that T_{x_0} is surjective (henceforth to be referred to as the Robinson condition; see [15]). We say that (cf. [16]) (T_{x_0}, F) satisfies the weak-Robinson condition at x_0 on $\mathbf{B}(x_0, r)$ if (1.15) holds: $-F(x_0) \in \mathrm{R}(T_{x_0})$, and

$$\mathrm{R}(F'(x)) \subseteq \mathrm{R}(T_{x_0}) \quad \text{for each } x \in \mathbf{B}(x_0, r). \tag{3.9}$$

For the case when F is C^2 on the involved closed ball $\overline{\mathbf{B}(x_0, r)}$, the notion of the γ-condition for nonlinear operators F in Banach spaces was first introduced by Wang [33] to study Smale's point estimate theory for operators which are not necessarily analytic. This notion was also used in [12] to improve the corresponding results in [26]. An extended version of this notion given below will be useful in the presence of convex processes (the equality in (3.10) is due to (2.10)).

Definition 3.1. Let $x_0 \in X$, $\gamma \geq 0$ and $0 < r \leq \frac{1}{\gamma}$. Let $F : \overline{\mathbf{B}(x_0, r)} \to \mathbb{R}$ be C^2. We say that (T_{x_0}, F) (or problem (1.1)) satisfies the weak γ-condition at x_0 on $\mathbf{B}(x_0, r)$ if

$$\|T_{x_0}^{-1} F''(x)\| \leq \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3} = \phi''(\|x-x_0\|) \quad \text{for each } x \in \mathbf{B}(x_0, r). \tag{3.10}$$

Note that (3.10) implies the following inclusion:

$$\mathrm{R}(F''(x)) \subseteq \mathrm{R}(T_{x_0}) \quad \text{for each } x \in \mathbf{B}(x_0, r). \tag{3.11}$$

Another consequence of (3.10) for the case when $\gamma = 0$ is given in the following remark.

Remark 3.1. Suppose that $\gamma = 0$ in Definition 3.1. Then, by (3.5), (3.10) and (1.13), one has that $0 \in T_{x_0}^{-1} F''(x)(u, v)$, namely $0 \in F''(x)(u, v) + C$ for all $u, v \in X$ and $x \in \mathbf{B}(x_0, r)$. Replacing v by $-v$, it follows that $F''(x)(u, v) \in C$ for all $u, v \in X$ and $x \in \mathbf{B}(x_0, r)$. Thus by continuity and since C is closed, we have

$$F''(x)(u, v) \in C \qquad \text{for all } u, v \in X \text{ and } x \in \overline{\mathbf{B}(x_0, r)}. \tag{3.12}$$

Proposition 3.1. *Suppose that X is reflexive. Let $x_0 \in X$, $\gamma \geq 0$ and $0 < r \leq \frac{1}{\gamma}$. Suppose that (T_{x_0}, F) satisfies the weak γ-condition at x_0 on $\mathbf{B}(x_0, r)$. Then the following assertions hold:*

(I) *(T_{x_0}, F) satisfies (3.9).*

(II) *Assume further that*

$$r \in (0, \frac{2-\sqrt{2}}{2\gamma}) \quad \textit{and} \quad x \in \mathbf{B}(x_0, r). \tag{3.13}$$

Then the following assertions hold.

(i) *For any $x' \in X$ satisfying $\|x - x'\| + \|x' - x_0\| < r$, we have*

$$T_{x_0}^{-1} \iint\limits_{[0,1]^2} [-sF''(x' + ts(x - x'))](x - x')^2 \,\mathrm{d}s\mathrm{d}t \neq \emptyset \tag{3.14}$$

and

$$\left\| T_{x_0}^{-1} \iint\limits_{[0,1]^2} [-sF''(x' + ts(x - x'))](x - x')^2 \,\mathrm{d}s\mathrm{d}t \right\| \leq \iint\limits_{[0,1]^2} s\phi''(\|x' - x_0\| + ts\|x - x'\|)\|x - x'\|^2 \,\mathrm{d}s\mathrm{d}t. \tag{3.15}$$

(ii) $\mathrm{D}(T_x^{-1} F'(x_0)) = X$,

$$\|T_{x_0}^{-1}(F'(x) - F'(x_0))\| < 1 \tag{3.16}$$

and

$$\|T_x^{-1} F'(x_0)\| \leq \left(2 - \frac{1}{(1-\gamma\|x-x_0\|)^2}\right)^{-1}. \tag{3.17}$$

(iii) *If* (1.15) *holds, then*

$$\mathfrak{D}(x) \neq \emptyset. \tag{3.18}$$

Proof. Let ϕ be defined as in (2.9) with given pair (γ, ξ) satisfying (2.7).

(I) Let $x \in \mathbf{B}(x_0, r)$, $u \in X$, and define continuous functions G and g on $[0, 1]$ respectively by

$$G(t) := F''(x_0 + t(x - x_0))(x - x_0, u) \quad \text{for each } t \in [0, 1]$$

and

$$g(t) := \phi''(t\|x - x_0\|)\|x_0 - x\|\|u\| \quad \text{for each } t \in [0, 1].$$

Then, by (3.10), we have, for each $t \in [0, 1[$,

$$\begin{aligned}\|T_{x_0}^{-1}G(t)\| &= \|T_{x_0}^{-1}F''(x_0 + t(x - x_0))(x - x_0, u)\| \\ &\leq \phi''(t\|x - x_0\|)\|x - x_0\|\|u\| \\ &= g(t).\end{aligned}$$

Hence, by [16, Lemma 2.1] (with $T_{x_0}^{-1}$ in place of T), one has that $T_{x_0}^{-1}\int_0^1 (F''(x_0+t(x-x_0))(x-x_0, u)\mathrm{d}t \neq \emptyset$. Since

$$F'(x)u - F'(x_0)u = \int_0^1 (F''(x_0 + t(x - x_0))(x - x_0, u)\mathrm{d}t,$$

it follows that $T_{x_0}^{-1}(F'(x)u - F'(x_0)u) \neq \emptyset$. Let $v \in T_{x_0}^{-1}(F'(x)u - F'(x_0)u)$. Then $F'(x_0)v \in F'(x)u - F'(x_0)u - C$. This means that

$$F'(x)(-u) \in F'(x_0)(-(u + v)) - C \subseteq \mathrm{R}(T_{x_0}).$$

Hence $\mathrm{R}(F'(x)) \subseteq \mathrm{R}(T_{x_0})$ and inclusion (3.9) is shown. The proof for (I) is complete.

(II) Let $x \in \mathbf{B}(x_0, r)$ and $x' \in X$ to satisfy $\|x - x'\| + \|x' - x_0\| < r$. In place of the functions G and g used in (I), we define continuous functions G and g on $[0, 1]^2$ respectively by

$$G(t, s) := -sF''(x' + ts(x - x'))(x - x')^2 \quad \text{for each } (t, s) \in [0, 1]^2$$

and

$$g(t, s) := \ s\phi''(\|x' - x_0\| + ts\|x - x'\|)\|x - x'\|^2 \quad \text{for each } (t, s) \in [0, 1]^2.$$

By (3.10), one has that, for any $s, t \in [0, 1]$,

$$\|T_{x_0}^{-1}G(t, s)\| \leq s\phi''(\|x' + ts(x - x') - x_0\|)\|x - x'\|^2 \leq s\phi''(\|x' - x_0\| + ts\|x - x'\|)\|x - x'\|^2 = g(t, s)$$

(noting that ϕ'' is increasing on $[0, \frac{1}{\gamma})$). Thus, Lemma 2.1 is applicable (to $T_{x_0}^{-1}$ in place of T) and assertion (i) follows.

To verify assertion (ii), we shall apply Lemma 3.1 to $A := F'(x) - F'(x_0)$. By (I), we have from (3.9) that $\mathrm{R}(F'(x_0) + A) \subseteq \mathrm{R}(T_{x_0})$. Next, since $x \in \mathbf{B}(x_0, r)$ and $r < \frac{2-\sqrt{2}}{2\gamma}$ by (3.13), we have from (2.10) that

$$\phi'(t) - \phi'(0) = \frac{1}{(1-\gamma t)^2} - 1 \quad \text{for each } t \in [0, \frac{1}{\gamma}). \tag{3.19}$$

In particular,

$$\phi'(\|x - x_0\|) - \phi'(0) < \phi'(r) - \phi'(0) = \frac{1}{(1-\gamma r)^2} - 1 \le \frac{1}{(1 - \frac{2-\sqrt{2}}{2})^2} - 1 = 1. \tag{3.20}$$

Since, for all $t \in [0, 1]$,

$$\|T_{x_0}^{-1}(F''(x_0 + t(x - x_0))\| \le \frac{2\gamma}{(1 - \gamma t\|x - x_0\|)^3} = \phi''(t\|x - x_0\|)$$

(thanks to the assumption of the weak γ-condition), we have by [16, Lemma 2.1] that

$$\begin{aligned} \|T_{x_0}^{-1}A\| &= \|T_{x_0}^{-1}(F'(x) - F'(x_0)) \\ &= \left\| T_{x_0}^{-1} \int_0^1 (F''(x_0 + t(x - x_0))(x - x_0)\mathrm{d}t \right\| \\ &\le \int_0^1 \phi''(t\|x - x_0\|)\|x - x_0\|\mathrm{d}t \\ &= \phi'(\|x - x_0\|) - \phi'(0) \end{aligned} \tag{3.21}$$

Hence, $\|T_{x_0}^{-1}A\| < 1$ by (3.20) and Lemma 3.1 is indeed applicable, and so (3.7) and (3.8) hold with $\tilde{T} := T_x$. In particular, (3.8), together with (3.21) and (3.19), implies that

$$\|T_x^{-1}F'(x_0)\| \le \frac{1}{1 - \|T_{x_0}^{-1}A\|} \le \frac{1}{1 - (\phi'(\|x - x_0\|) - \phi'(0))} = \left(2 - \frac{1}{(1 - \gamma\|x - x_0\|)^2}\right)^{-1}.$$

This, together with (3.7) shows assertion (ii).

Below we show assertion (iii). To do this, assume that (1.15) holds. By (i) and (ii) (applied to $[x_0, x]$ in place of $[x', x]$), we have

$$\mathrm{D}(T_x^{-1}F'(x_0)) = X \tag{3.22}$$

and

$$T_{x_0}^{-1} \iint_{[0,1]^2} [-sF''(x_0 + ts(x - x_0))](x - x_0)^2 \,\mathrm{d}s\mathrm{d}\,t \ne \emptyset,$$

so

$$T_x^{-1}F'(x_0)T_{x_0}^{-1} \iint_{[0,1]^2} [-sF''(x_0 + ts(x - x_0))](x - x_0)^2 \,\mathrm{d}s\mathrm{d}t \ne \emptyset. \tag{3.23}$$

Consequently, by (3.1) and (3.22), we get

$$T_x^{-1} \iint\limits_{[0,1]^2} [-sF''(x_0 + ts(x - x_0))](x - x_0)^2 \,\mathrm{d}s\mathrm{d}t \neq \emptyset, \tag{3.24}$$

and

$$(T_x^{-1} F'(x_0))(x_0 - x) \neq \emptyset. \tag{3.25}$$

Noting by the fundamental theorem of calculus that

$$\begin{aligned} F(x_0) - F(x) &= -\int_0^1 F'(x_0 + s(x - x_0))\,(x - x_0)\mathrm{d}s \\ &= \iint\limits_{[0,1]^2} [-sF''(x_0 + ts(x - x_0))](x - x_0)^2 \,\mathrm{d}s\mathrm{d}\,t + F'(x_0)(x_0 - x), \end{aligned}$$

and since T_x is a convex process, it follows from (3.24) and (3.25) that

$$T_x^{-1}(F(x_0) - F(x)) \neq \emptyset. \tag{3.26}$$

Similarly, by (3.22), (1.15) and (3.1) again, we have that

$$\emptyset \neq T_x^{-1} F'(x_0) T_{x_0}^{-1}(-F(x_0)) \subseteq T_x^{-1}(-F(x_0)). \tag{3.27}$$

From the convex process property,

$$T_x^{-1}(-F(x)) \supseteq T_x^{-1}(-F(x_0)) + T_x^{-1}(F(x_0) - F(x)), \tag{3.28}$$

we make use of (3.26) and (3.27) to conclude that $T_x^{-1}(-F(x)) \neq \emptyset$, that is, (3.18) holds (see (3.2)) and assertion (iii) is shown. The proof is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First we deal with the case when $\gamma = 0$. By (1.15), (3.2), and (1.17), we have that $\mathfrak{D}(x_0) \neq \emptyset$ and $\|\mathfrak{D}(x_0)\| = \xi = r^*$. Since $\mathfrak{D}(x_0)$ is closed and X is reflexive, it follows that $x_1 = x_0 + d_1$ for some $d_1 \in \mathfrak{D}(x_0)$ with $\|d_1\| = \xi$. By (3.2), (1.10) and (1.13), one has that

$$F(x_0) + F'(x_0)(x_1 - x_0) \in C.$$

Moreover, by Definition 3.1 and Remark 3.1 (applied to r^* in place of r, thanks to the given γ-condition assumption), we have that

$$F''(x_0 + t(x_1 - x_0)(x_1 - x_0)^2(1 - t) \in C \quad \text{ for each } t \in [0, 1].$$

Hence, since C is a closed, the fundamental theorem of calculus implies that $F(x_1) \in C$ because

$$F(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) + \int_0^1 F''(x_0 + t(x_1 - x_0)(x_1 - x_0)^2(1 - t)\mathrm{d}t \in C + C = C.$$

Thus, $x^* := x_1 = x_n$ for all $n \in \mathbb{N}$ solving (1.1); clearly $\{x_n\} \subseteq \overline{\mathbf{B}(x_0, r^*)}$, (1.22) and (1.23) hold trivially in this case.

Henceforth we consider the case when $\gamma > 0$. Let $\{t_n\}$ be the sequence defined by (2.19). Then, by (2.12)–(2.15) and Lemma 2.2, we have that

$$\xi = t_1 \le t_n < r^* \le r_0 = \frac{2-\sqrt{2}}{2\gamma} \quad \text{for all } n \ge 1. \tag{3.29}$$

We will show that Algorithm $\mathbf{A}(x_0)$ generates at least one sequence $\{x_n\}$ and, for any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$, the following assertions hold for each n:

$$\|x_{n+1} - x_n\| \le (t_{n+1} - t_n)\left(\frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}}\right)^2, \tag{3.30}$$

$$\|x_n - x_{n-1}\| \le t_n - t_{n-1}, \tag{3.31}$$

and

$$F(x_{n-1}) + F'(x_{n-1})(x_n - x_{n-1}) \in C. \tag{3.32}$$

Granting these, one has that, for all $m \ge n \ge 0$,

$$\|x_n - x_m\| \le t_m - t_n \to r^* - t_n$$

as $m \to \infty$. In particular, $x^* := \lim_{n\to\infty} x_n$ exists in X and satisfies that

$$\|x_n - x^*\| \le r^* - t_n. \tag{3.33}$$

Passing to the limit in (3.32), one has that $F(x^*) \in C$, so x^* solves (1.1).

It remains to verify that Algorithm $\mathbf{A}(x_0)$ generates at least one sequence $\{x_n\}$, and that any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$ satisfies (3.30)–(3.32). We do by mathematical induction. Let us first note that Proposition 3.1 is applicable (applied to r^* in place of r, thanks to the given weak γ-condition assumption and (3.29)). Hence, by (1.15), (3.2) and (3.18), we have that

$$\mathfrak{D}(x) = T_x^{-1}(-F(x)) \ne \emptyset \quad \text{for each } x \in \mathbf{B}(x_0, r^*). \tag{3.34}$$

In particular,

$$\mathfrak{D}(x_0) = T_{x_0}^{-1}(-F(x_0)) \ne \emptyset. \tag{3.35}$$

This, together with (1.16) and (2.15), implies that

$$\mathrm{d}(0, \mathfrak{D}(x_0)) = \|T_{x_0}^{-1}(-F(x_0))\| = \xi = t_1 - t_0. \tag{3.36}$$

Since X is reflexive, it follows that there exists $d_1 \in \mathfrak{D}(x_0)$ such that $\|d_1\| = d(0, \mathfrak{D}(x_0))$ and so x_1 is generated. Thus, $F(x_0) + F'(x_0)(x_1 - x_0) \in C$ by Algorithm $\mathbf{A}(x_0)$ and so (3.32) holds for $n = 1$. Furthermore, by (3.36), one has that

$$\|d_1\| = d(0, \mathfrak{D}(x_0)) = t_1 - t_0,$$

i.e., $\|x_1 - x_0\| = t_1 - t_0$. This shows that (3.31) holds for $n = 1$.

Assume that (3.31) and (3.32) hold for all $n = 1, 2, \ldots, k$. We will complete our induction by showing that (3.30) also holds for $n = k$, and that (3.31) and (3.32) hold for each $n = k + 1$. To do this, note first that

$$\|x_k - x_0\| \le \sum_{i=1}^{k} \|x_i - x_{i-1}\| \le \sum_{i=1}^{k}(t_i - t_{i-1}) = t_k \tag{3.37}$$

and

$$\|x_{k-1} - x_0\| \le t_{k-1} \le t_k \tag{3.38}$$

because $\{t_n\}$ is increasing (see Lemma 2.2). In particular, $x_k \in \mathbf{B}(x_0, r^*)$ and so, by (3.17) (applied to r^* in place of r, again thanks to the given weak γ-condition assumption and (3.29)) and (3.34), one has

$$\|T_{x_k}^{-1} F'(x_0)\| \le \left(2 - \frac{1}{(1 - \gamma\|x_k - x_0\|)^2}\right)^{-1} \quad \text{and} \quad \mathfrak{D}(x_k) \ne \emptyset. \tag{3.39}$$

In particular, in view of Algorithm $\mathbf{A}(x_0)$, x_{k+1} can be generated (as we noted for the case when $k = 0$) and (3.32) holds for $n = k + 1$. We further claim that

$$\emptyset \ne (T_{x_k}^{-1} F'(x_0)) T_{x_0}^{-1}[-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})] \subseteq \mathfrak{D}(x_k). \tag{3.40}$$

Noting $D(T_{x_k}^{-1} F'(x_0)) = X$ by assertion (ii) of Proposition 3.1(II) (applied to x_k in place of x, recalling that Proposition 3.1 is applicable as noted earlier), to prove the above nonemptiness assertion, it is sufficient to show that

$$T_{x_0}^{-1}[-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})] \ne \emptyset. \tag{3.41}$$

Recall that $\|x_{k-1} - x_0\| \le t_{k-1}$ by (3.38) and $\|x_k - x_{k-1}\| \le t_k - t_{k-1}$ by assumptions earlier. Since

$$-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1}) = \iint\limits_{[0,1]^2} [-\tau F''(x_{k-1} + \tau t(x_k - x_{k-1}))](x_k - x_{k-1})^2 \, \mathrm{d}\tau \mathrm{d}t,$$

(3.41) follows from Proposition 3.1 II(i) (applied to $[x_{k-1}, x_k]$ in place of $[x', x]$ and noting that $\|x_k - x_{k-1}\| + \|x_{k-1} - x_0\| \le t_k < r^*$). This and (3.15) imply that

$$\begin{aligned}
&\|T_{x_0}^{-1}[-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})]\| \\
&= \left\| T_{x_0}^{-1} \iint\limits_{[0,1]^2} [-\tau F''(x_{k-1} + \tau t(x_k - x_{k-1}))](x_k - x_{k-1})^2 \, \mathrm{d}\tau \mathrm{d}t \right\| \\
&\le \iint\limits_{[0,1]^2} \tau \phi''(\|x_{k-1} - x_0\| + t\tau\|x_k - x_{k-1}\|)\|x_k - x_{k-1}\|^2 \, \mathrm{d}\tau \mathrm{d}t \\
&\le \iint\limits_{[0,1]^2} \tau \phi''(t_{k-1} + t\tau(t_k - t_{k-1}))(t_k - t_{k-1})^2 \, \mathrm{d}\tau \mathrm{d}t \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2.
\end{aligned} \tag{3.42}$$

An elementary calculation shows that

$$\phi(t_k) = \iint\limits_{[0,1]^2} \tau\phi''(t_{k-1} + t\tau(t_k - t_{k-1}))(t_k - t_{k-1})^2 \,\mathrm{d}\tau\mathrm{d}t,$$

and it follows from (3.42) that

$$\left\|T_{x_0}^{-1}[-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})]\right\| \le \phi(t_k) \cdot \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2. \tag{3.43}$$

We next show the inclusion in (3.40). Let $z := -F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})$ and $d \in (T_{x_k}^{-1}F'(x_0))T_{x_0}^{-1}(z)$, that is, $d \in (T_{x_k}^{-1}F'(x_0))u$ for some $u \in T_{x_0}^{-1}(z)$. We have to show that $d \in \mathfrak{D}(x_k)$. Note that $F'(x_k)d \in F'(x_0)u + C$ and $F'(x_0)u \in z + C$, so $F'(x_k)d \in z + C + C = z + C$, since C is a convex cone. Making use of the definition of z, it follows that

$$F(x_k) + F'(x_k)d \in F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1}) + C \subseteq C + C = C,$$

where the inclusion holds as (3.32) holds for $n = k$ by assumption, that is $d \in \mathfrak{D}(x_k)$ as we want to show. Therefore, (3.40) is valid and it follows from (3.39) and (3.43) that

$$\begin{aligned} d(0, \mathfrak{D}(x_k)) &\le \left\| \left(T_{x_k}^{-1}F'(x_0)\right) T_{x_0}^{-1}[-F(x_k) + F(x_{k-1}) + F'(x_{k-1})(x_k - x_{k-1})]\right\| \\ &\le \left(2 - \frac{1}{(1-\gamma t_k)^2}\right)^{-1} \cdot \phi(t_k) \cdot \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 \\ &= -\frac{\phi(t_k)}{\phi'(t_k)} \cdot \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 \\ &= (t_{k+1} - t_k)\left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2, \end{aligned} \tag{3.44}$$

where the last equality is true by (2.14), while the first equality holds because of the first equality of (2.10):

$$\left(2 - \frac{1}{(1-\gamma t_k)^2}\right)^{-1} = -\phi'(t_k)^{-1}. \tag{3.45}$$

Hence,

$$\|x_{k+1} - x_k\| = \|d_k\| \le d(0, \mathfrak{D}(x_k)) \le (t_{k+1} - t_k)\left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2, \tag{3.46}$$

showing that (3.30) holds for $n = k$. Since (3.31) holds for $n = k$ by inductional assumptions, this implies that $\|x_{k+1} - x_k\| \le t_{k+1} - t_k$, namely (3.31) holds for $n = k+1$. This completes our mathematical induction argument and the proof is complete. □

4. Approximate solutions and the extended Smale α-theory

In his fundamental work on point estimate theory regarding Newton's method for solving the nonlinear analytic equation $F(x) = 0$, where $F : \Omega \subseteq X \to Y$ is an analytic function, Smale (cf. [28–30]; see also [1]) made an important use of his assumption that $F'(x_0)$ is nonsingular (at starting point $x_0 \in \Omega$). In the course of his study, a key notion is the quantity $\gamma(F, x_0) \in \mathbb{R}$, defined by

$$\gamma(F, x_0) := \sup_{k\ge 2} \left\|\frac{F'(x_0)^{-1}F^{(k)}(x_0)}{k!}\right\|^{\frac{1}{k-1}}. \tag{4.1}$$

To extend the theory of the Smale point estimate to cover the case of $F'(x_0)$ being not necessarily nonsingular, Smale and Shub introduced in [26] assumption (1.9):

$$F'(x_0) \text{ is surjective,} \tag{4.2}$$

and replaced $F'(x_0)^{-1}$ in (4.1) by $F'(x_0)^\dagger$, the Moore–Penrose inverse of $F'(x_0)$, but only in the case when X and Y are finite-dimensional. The present section is devoted to an attempt addressing similar issues for problem (1.1); in particular, we give the proofs of Theorem 1.2 and Corollary 1.1. We make the following assumption (as in the Smale theory):

- F is analytic on Ω.

Moreover, we define a quantity $\gamma_{(F,C)}(x_0) \in \mathbb{R} \cup \{+\infty\}$ by

$$\gamma_{(F,C)}(x_0) := \sup_{k\geq 2} \left\| \frac{T_{x_0}^{-1} F^{(k)}(x_0)}{k!} \right\|^{\frac{1}{k-1}}, \tag{4.3}$$

where $\|T_{x_0}^{-1} F^{(k)}(x_0)\|$ is defined as in (1.18). Similar to the weak γ-condition, let us say that (T_{x_0}, F) (or problem (1.1)) satisfies the weak-Smale condition at x_0 if

$$\gamma_{(F,C)}(x_0) < +\infty; \tag{4.4}$$

this in particular implies that

$$\mathrm{R}(F^{(k)}(x_0)) \subseteq \mathrm{R}(T_{x_0}) \quad \text{ for each } k = 2, 3, \cdots. \tag{4.5}$$

Remark 4.1. Suppose that $F'(x_0)$ is nonsingular, or that X and Y are finite-dimensional. If $C = \{0\}$ and (4.2) holds, then, by the definition of the Moore–Penrose inverse, one has that $\|T_{x_0}^{-1} F^{(k)}(x_0)\| = \|F'(x_0)^\dagger F^{(k)}(x_0)\|$ for each $k \in \mathbb{N}$, and so $\gamma_{(F,C)}(x_0) = \gamma(F, x_0)$.

For our discussion in Proposition 4.1, it is convenient to introduce a notion of the sum for a sequence of convex processes. Let $\{T_k\}$ be a sequence of convex processes form X to Y. For each $x \in X$, define $T(x)$ to be the set of all points $u \in Y$ for which there exists a sequence $\{u_k\} \subseteq Y$ with $u_k \in T_k(x)$ for each k such that $u = \sum_{k=1}^{\infty} u_k$. Note that T is a convex process with $\mathrm{D}(T) \subseteq \bigcap_{k=1}^{\infty} \mathrm{D}(T_k)$. It would be convenient to denote this T more suggestively by $\sum_{k=1}^{\infty} T_k$ and call it the sum of $\{T_k\}$.

Lemma 4.1. *Suppose that* $\mathrm{D}(T_k) = X$ *for each* k *and* $\sum_{k=1}^{\infty} \|T_k\| < +\infty$. *Let* $T = \sum_{k=1}^{\infty} T_k$. *Then* $\mathrm{D}(T) = X$ *and* $\|T\| \leq \sum_{k=1}^{\infty} \|T_k\|$.

Proof. Let $z \in X$ and let $\epsilon > 0$. Then there exists $u_k \in T_k(z)$ such that $\|u_k\| \leq (\|T_k\| + \frac{\epsilon}{2^k})\|z\|$. Hence, $\sum_{k=1}^{\infty} u_k$ converges, say to u. Then $u \in T(z)$ and $z \in \mathrm{D}(T)$. Furthermore, we have that

$$\|u\| \le \left(\sum_{k=1}^{\infty} \|T_k\|\right) \|z\| + \epsilon\|z\|.$$

Therefore, $\|T\| \le \sum_{k=1}^{\infty} \|T_k\|$ and the proof is complete. □

Proposition 4.1. *Let $x_0 \in X$ and $\gamma := \gamma_{(F,C)}(x_0)$ (see (4.3)). Suppose that (T_{x_0}, F) satisfies the weak-Smale condition at x_0, that is $\gamma < +\infty$. Then it satisfies the weak γ-condition at x_0 on $\mathbf{B}(x_0, \frac{1}{\gamma})$.*

Proof. Let $x \in \mathbf{B}(x_0, \frac{1}{\gamma})$. In view of Definition 3.1, we only need to verify that

$$\|T_{x_0}^{-1} F''(x)\| \le \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3}. \tag{4.6}$$

To do this, we first consider a function $\tilde{\gamma}_F : X \to \mathbb{R}$ defined by

$$\tilde{\gamma}_F(y) := \sup_{k \ge 2} \left\| \frac{F^{(k)}(y)}{k!} \right\|^{\frac{1}{k-1}} \quad \text{for each } y \in X$$

($\tilde{\gamma}_F(y) < +\infty$ for each $y \in X$ because F is analytic). We claim that $\tilde{\gamma}_F$ is continuous on X. Since it is easy to verify it is lower semicontinuous (as it is the super function of the continuous functions $\{\left\|\frac{F^{(k)}(\cdot)}{k!}\right\|^{\frac{1}{k-1}}\}$), we only need to check the upper semicontinuity at an arbitrary point $y_0 \in X$. To do this, let $y \in \mathbf{B}(y_0, \frac{1}{\tilde{\gamma}_F(y_0)})$. We will make use of a well-known identity (cf. [1, p. 150])

$$\sum_{j=0}^{\infty} \frac{(k+j)!}{k!\,j!} t^j = \frac{1}{(1-t)^{k+1}} \quad \text{for each } t \in [-1, 1) \text{ and } k = 0, 1, \cdots \tag{4.7}$$

to show that

$$\tilde{\gamma}_F(y) \le \frac{\tilde{\gamma}_F(y_0)}{\left(1 - \tilde{\gamma}_F(y_0)\|y - y_0\|\right)^3} \tag{4.8}$$

(so in passing to the limit as $y \to y_0$, we have that $\limsup_{y \to y_0} \tilde{\gamma}_F(y) \le \tilde{\gamma}_F(y_0)$, that is $\tilde{\gamma}_F(\cdot)$ is upper semicontinuous at y_0). Indeed, by Taylor formula,

$$F^{(k)}(y) = \sum_{j=0}^{\infty} \frac{F^{(k+j)}(y_0)}{j!}(y - y_0)^j \quad \text{for any } y_0 \in X \text{ and } y \in \mathbf{B}(y_0, \tfrac{1}{\tilde{\gamma}_F(y_0)}). \tag{4.9}$$

Since $\left\|\frac{F^{(k)}(y_0)}{k!}\right\| \le \tilde{\gamma}_F(y_0)^{k-1}$ for each $k \ge 2$, it follows that

$$\begin{aligned}
\left\|\frac{F^{(k)}(y)}{k!}\right\| &\le \sum_{j=0}^{\infty} \frac{\|F^{(k+j)}(y_0)\|}{k!\ j!} \|y - y_0\|^j \\
&\le \sum_{j=0}^{\infty} \frac{(k+j)!\,\tilde{\gamma}_F(y_0)^{k+j-1}}{k!\ j!} \|y - y_0\|^j \\
&= \tilde{\gamma}_F(y_0)^{k-1} \sum_{j=0}^{\infty} \frac{(k+j)!}{k!\ j!} (\tilde{\gamma}_F(y_0)\|y - y_0\|)^j \\
&= \frac{\tilde{\gamma}_F(y_0)^{k-1}}{\left(1 - (\tilde{\gamma}_F(y_0)\|y - y_0\|)\right)^{k+1}},
\end{aligned}$$

where the last equality holds by (4.7) (applied to $\tilde{\gamma}_F(y_0)\|y-y_0\|$ in place of t, as $y \in \mathbf{B}(y_0, \frac{1}{\tilde{\gamma}_F(y_0)})$). Therefore, (4.8) holds as

$$\tilde{\gamma}_F(y) = \sup_{k\geq 2}\left\|\frac{F^{(k)}(y)}{k!}\right\|^{\frac{1}{k-1}} \leq \sup_{k\geq 2}\frac{\tilde{\gamma}_F(y_0)}{\left(1-\tilde{\gamma}_F(y_0)\|y-y_0\|\right)^{\frac{k+1}{k-1}}} = \frac{\tilde{\gamma}_F(y_0)}{\left(1-\tilde{\gamma}_F(y_0)\|y-y_0\|\right)^3},$$

where the last equality holds because the sequence $\{a^{\frac{k+1}{k-1}}\}$ is increasing for any $a \in (0,1)$. Now we know that $\tilde{\gamma}_F$ is continuous and so there exists a positive constant λ such that $\tilde{\gamma}_F(y) \leq \lambda$ for all $y \in [x_0, x]$, the line-segment with end points x_0 and x. Take a natural number p such that $\frac{\lambda\|x-x_0\|}{p} < 1$. Subdivide $[x_0, x]$ into p subsegments with equal length determined by the consecutive points $x_0 < x_1 < \cdots < x_{p-1} < x_p = x$. Thus

$$\|x_i - x_{i-1}\| < \frac{1}{\tilde{\gamma}_F(x_i)} \quad \text{for each } i = 1, \ldots, p. \tag{4.10}$$

Consequently, by (4.9),

$$F^{(k)}(x_i) = \sum_{j=0}^{\infty}\frac{1}{j!}F^{(k+j)}(x_{i-1})(x_i - x_{i-1})^j \quad \text{for each } i = 1, \ldots, p. \tag{4.11}$$

We claim that, for each $i = 0, 1, \ldots, p$,

$$\left\|\frac{T_{x_0}^{-1}F^{(k)}(x_i)}{k!}\right\| \leq \frac{\gamma^{k-1}}{(1-\gamma\|x_i - x_0\|)^{k+1}} \quad \text{for each } k = 1, 2, \cdots. \tag{4.12}$$

(This in particular implies (4.6) holds as $x = x_p$.) Indeed, this is certainly true when $i = 0$ as, by definition, $\|T_{x_0}^{-1}F'(x_0)\| \leq 1$ and $\left\|\frac{T_{x_0}^{-1}F^{(k)}(x_0)}{k!}\right\| \leq \gamma^{k-1}$ for all $k = 2, \cdots$. Inductively, we assume that (4.12) holds for some $m-1 \leq p$:

$$\left\|\frac{T_{x_0}^{-1}F^{(k)}(x_{m-1})}{k!}\right\| \leq \frac{\gamma^{k-1}}{(1-\gamma\|x_{m-1} - x_0\|)^{k+1}} \quad \text{for each } k = 1, 2, \cdots. \tag{4.13}$$

To establish our claim, we need only to show that (4.12) holds when i is replaced by m. To do this, let $k \in \mathbb{N}$ be arbitrary but fixed and let $z_1, z_2 \cdots, z_{k-1} \in X$ be of norm no more than 1. It suffices to show that

$$\left\|\frac{T_{x_0}^{-1}F^{(k)}(x_m)(z_1\, z_2 \cdots z_{k-1})}{k!}\right\| \leq \frac{\gamma^{k-1}}{(1-\gamma\|x_m - x_0\|)^{k+1}}. \tag{4.14}$$

To proceed, let $j \in \mathbb{N} \cup \{0\}$ and let $T_j : X \to X$ be the convex process defined by

$$T_j = \frac{1}{j!}T_{x_0}^{-1}F^{(k+j)}(x_{m-1})((x_m - x_{m-1})^j z_1\, z_2 \cdots z_{k-1}). \tag{4.15}$$

Then $\mathrm{D}(T_j) = X$ as (4.5) holds by the assumed weak-Smale condition. Noting $((x_m - x_{m-1})^j, z_1, z_2, \cdots, z_{k-1}) \in (X)^{k+j-1}$ and $z_1, z_2 \cdots, z_{k-1} \in X$ are of norm no more than 1, it follows from (4.13) (applied to $k+j$ in place of k) that

$$\|T_j\| \leq \frac{(k+j)!\gamma^{k+j-1}\|x_m - x_{m-1}\|^j}{j!\,(1-\gamma\|x_{m-1} - x_0\|)^{k+j+1}}. \tag{4.16}$$

Let a_j denote the expression on the right-hand side of (4.16), i.e.,

$$a_j := \frac{(k+j)!\gamma^{k+j-1}\|x_m - x_{m-1}\|^j}{j!\,(1-\gamma\|x_{m-1}-x_0\|)^{k+j+1}} = \frac{k!\,\gamma^{k-1}}{(1-\gamma\|x_{m-1}-x_0\|)^{k+1}}\left[\frac{(k+j)!}{k!\,j!}\left(\frac{\gamma\|x_m-x_{m-1}\|}{1-\gamma\|x_{m-1}-x_0\|}\right)^j\right].$$

Since x_0, x_{m-1} and x_m are in the same line-segment (so $\|x_m - x_0\| = \|x_m - x_{m-1}\| + \|x_{m-1} - x_0\|$) and applying (4.7) (to $\frac{\gamma\|x_m-x_{m-1}\|}{1-\gamma\|x_{m-1}-x_0\|}$ in place of t), one has that

$$\sum_{j=0}^{\infty} a_j = \frac{k!\,\gamma^{k-1}}{(1-\gamma\|x_{m-1}-x_0\|)^{k+1}} \cdot \frac{1}{\left(1-\frac{\gamma\|x_m-x_{m-1}\|}{1-\gamma\|x_{m-1}-x_0\|}\right)^{k+1}} = \frac{k!\,\gamma^{k-1}}{(1-\gamma\|x_m-x_0\|)^{k+1}} < +\infty.$$

Therefore, $\sum_{j=0}^{\infty}\|T_j\| < +\infty$, and it follows from Lemma 4.1 that $T := \sum_{j=0}^{\infty} T_j$ has the properties

$$\mathrm{D}(T) = X \quad \text{and} \quad \|T\| \le \frac{k!\,\gamma^{k-1}}{(1-\gamma\|x_m-x_0\|)^{k+1}}. \tag{4.17}$$

We will show that

$$T(z) \subseteq T_{x_0}^{-1}\big(F^k(x_m)z_1\, z_2 \cdots z_{k-1}\big)(z) \quad \text{for each } z \in X. \tag{4.18}$$

Granting this, we then have from (4.17) that (4.14) holds and so (4.12) holds when i is replaced by m.

Thus it remains to show that (4.18). To verify this, let $z \in X$ and $u \in T(z)$. By definition of T, we represent $u = \sum_{j=0}^{\infty} u_j$ with each $u_j \in T_j(z)$. By definition of T_j, we have that

$$F'(x_0)u_j \in \frac{1}{j!}F^{(k+j)}(x_{m-1})((x_m - x_{m-1})^j z_1\, z_2 \cdots z_{k-1}z) + C \quad \text{for each } k = 2, 3, \cdots,$$

and so for each $n \in \mathbb{N}$,

$$F'(x_0)\left(\sum_{j=0}^{n} u_j\right) \in \left(\sum_{j=0}^{n} \frac{F^{(k+j)}(x_{m-1})((x_m - x_{m-1})^j z_1\, z_2 \cdots z_{k-1}z)}{j!}\right) + C.$$

Noting from (4.11) (applied to m in place of i) that

$$\lim_{n\to\infty}\sum_{j=0}^{n} \frac{F^{(k+j)}(x_{m-1})((x_m - x_{m-1})^j z_1\, z_2 \cdots z_{k-1}z)}{j!} = F^{(k)}(x_m)(z_1\, z_2 \cdots z_{k-1}z),$$

we then have $F'(x_0)u \in F^{(k)}(x_m)(z_1\, z_2 \cdots z_{k-1}z) + C$, verifying (4.18). The proof is complete. □

Combining Theorem 1.1 and Proposition 4.1, we obtain the following corollary. Recall that ξ is defined by (1.16).

Corollary 4.1. *Let $\Omega \supseteq \mathbf{B}(x_0, \frac{1}{\gamma})$. Suppose that X is reflexive, and (1.15) holds: $-F(x_0) \in \mathrm{R}(T_{x_0})$. Suppose further that (1.20) and (1.26) hold. Then x_0 is an implementable starting point for Algorithm $\mathbf{A}(x_0)$ and any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$ converges to a solution x^* of (1.1) satisfying (1.22) and (1.23).*

In connection with the issue of solving the operator equation $F = 0$ by Newton's method, one of the key notions is the one due to Smale: approximate zeros. There are several different (nonequivalent) versions of approximate zeros. The first one was introduced by him in [27] but was replaced by a new one in [29,28] (see also [1] and [30]) as the later describes the property of quadratic convergence of the sequences generated by Newton's method and is more convenient for applying the theory to the study of the computational complexity. The definition of his second one is: x_0 is said to be an approximate zero of F if Newton's method (1.4) is well-defined and

$$\|x_{n+1} - x_n\| \le \left(\tfrac{1}{2}\right)^{2^{n-1}} \|x_1 - x_0\|$$

for each $n = 1, 2, \dots$. The following version due to Wang [31] is similar but slightly stronger:

Definition 4.1. A starting point $x_0 \in \Omega$ is called an approximate zero of F if it is an implementable starting point for Newton's method (1.4) and the sequence $\{x_n\}$ generated by Newton's method (1.4) satisfies the Smale error estimate:

$$\|x_{n+1} - x_n\| \le \left(\frac{1}{2}\right)^{2^{n-1}} \|x_n - x_{n-1}\| \quad \text{for all } n = 1, 2, \dots. \tag{4.19}$$

Another variant of the notion of approximate zeros was introduced in [4] and is defined by

$$\|F'(x_0)^{-1}F(x_n)\| \le \left(\frac{1}{2}\right)^{2^{n-1}} \|F'(x_0)^{-1}F(x_{n-1})\| \quad \text{for all } n = 1, 2, \dots \tag{4.20}$$

which turns out, as shown in [31], to be equivalent to that given in Definition 4.1. Along the line of (4.19) and (4.20), the corresponding notion is extended in [14] to the Gauss–Newton method for singular equations in Banach spaces.

Similar to Definition 4.1, we adopt the following definition of approximate zeros to the extended Newton method for solving problem (1.1).

Definition 4.2. A point $x_0 \in \Omega$ is called an approximate solution of (1.1) for the extended Newton method Algorithm $\mathbf{A}(x_0)$ if x_0 is an implementable starting point for Algorithm $\mathbf{A}(x_0)$ and any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$ satisfies the Smale error estimate (4.19).

Recall that ξ and r^* are defined by (1.16) and (1.17), respectively. Clearly, Theorem 1.2 is a direct consequence of Proposition 4.1 and the following theorem.

Theorem 4.1. *Suppose that X is reflexive. Let $\Omega \supseteq \overline{\mathbf{B}(x_0, r^*)}$, and F be C^2 on $\overline{\mathbf{B}(x_0, r^*)}$ such that (1.15) and (1.25) hold. Suppose that (T_{x_0}, F) satisfies the weak γ-condition at x_0 on $\mathbf{B}(x_0, r^*)$. Then, assumptions (1.20) and (1.21) hold, and x_0 is an approximate solution of (1.1). Moreover, any sequence $\{x_n\}$ generated by Algorithm $\mathbf{A}(x_0)$ is contained in $\overline{\mathbf{B}(x_0, r^*)}$.*

Proof. As noted before, we may assume that $\gamma > 0$. By the elementary inequality $\frac{13-3\sqrt{17}}{4\gamma} < \frac{3-2\sqrt{2}}{\gamma}$, it is evident that, if (1.25) holds then (1.20) holds and $q \le \frac{1}{2}$ (noting (2.18)). Thus, the present theorem follows from Theorem 1.1 and the proof is complete. □

In connection with Corollary 1.1, it would be convenient to recall some basic facts regarding generalized inverses. We consider X and Y as before: X is a Hilbert space and Y is a Banach space. For a nonempty closed convex W of X and $x \in X$, let $\Pi_W(x)$ denote the best approximation w_0 to x from W, namely

$w_0 \in W$ and $\|w_0 - x\| \le \|w - x\|$ for any $w \in W$. Recall the notation $\|W\| = \inf_{w\in W} \|w\|$. A closed subspace Z of Y is said to be complemented in Y if it is the range of a bounded linear projection Q from Y to Z:

$$Z = \{Qy : y \in Y\}. \tag{4.21}$$

The first result in the following lemma is known (cf. [21]):

Lemma 4.2. *Let $A : Y \to X$ be a bounded linear operator such that $\mathrm{R}(A)$ is complemented in Y. Then there exists a bounded linear operator $A^+ : Y \to X$ such that*

$$A\,A^+\,A = A, \quad A^+\,A\,A^+ = A^+, \quad A^+\,A = \Pi_{\ker A^\perp} \quad \textit{and} \quad A\,A^+ = Q, \tag{4.22}$$

where Q is a bounded linear projection Q from Y to $\mathrm{R}(A)$, and $\ker A^\perp$ denotes the orthogonal complement of the kernel of A. Moreover, for each $y \in \mathrm{R}(A)$, one has that

$$A^+y \in A^{-1}(y) \quad \textit{and} \quad \|A^+y\| = \|A^{-1}y\|, \tag{4.23}$$

that is

$$A^+y = \Pi_{A^{-1}y}0. \tag{4.24}$$

Proof. We only need to prove the second result, that is (4.23). Let $y \in \mathrm{R}(A)$. Then $A(A^+y) = Qy = y$ by (4.22), and so the first assertion in (4.23) holds. Further, if $x \in A^{-1}y$, then $A^+y = A^+(Ax) = \Pi_{\ker A^\perp} x$ by (4.22), and so $\|A^+y\| \le \|x\|$, proving the second assertion. □

Now we turn to the proof of Corollary 1.1.

Proof of Corollary 1.1. Thanks to the giving assumptions on our starting point x_0, one can apply Lemma 4.2 to $F'(x_0)$ (in place of A) that, for each $k \in \mathbb{N}$, the norm of the convex process $F'(x_0)^{-1}F^{(k)}(x_0)$ has the following estimate:

$$\|F'(x_0)^{-1}F^{(k)}(x_0)\| = \|F'(x_0)^+F^{(k)}(x_0)\| \le \|F'(x_0)^+\|\|F^{(k)}(x_0)\|. \tag{4.25}$$

Letting $\eta := \sup_{m\in\mathbb{N}} \|F'(x_0)^+\|^{\frac{1}{m}}$, it follows that

$$\gamma := \sup_{k\ge 2} \left\| \frac{F'(x_0)^+F^{(k)}(x_0)}{k!} \right\|^{\frac{1}{k-1}} \le \eta \sup_{k\ge 2} \left\| \frac{F^{(k)}(x_0)}{k!} \right\|^{\frac{1}{k-1}} < +\infty, \tag{4.26}$$

as F is analytic at x_0. Now let $C := \{0\}$ and consider any $x \in \Omega$. Then one has that

$$T_x = F'(x) \quad \text{and} \quad T_x^{-1}(y) = F'(x)^{-1}(y) \quad \text{for any } y \in Y. \tag{4.27}$$

Hence we have

$$T_{x_0}^{-1}F^{(k)}(x_0) = F'(x_0)^{-1}F^{(k)}(x_0) \quad \text{for each } k \ge 2,$$

and it follows from (4.26) that

$$\sup_{k\geq 2}\left\|\frac{T_{x_0}^{-1}F^{(k)}(x_0)}{k!}\right\|^{\frac{1}{k-1}}=\gamma<+\infty.$$

Together with the other assumptions of the corollary, one can apply Theorem 1.2 to conclude that x_0 is an approximate solution of (1.1) with the associated sequence $\{x_n\}$ (contained in $\overline{\mathbf{B}}(x_0,r^*)$) generated by Algorithm $\mathbf{A}(x_0)$: For each $n\in\mathbb{N}\cup\{0\}$,

$$x_{n+1}-x_n=\Pi_{\mathcal{D}(x_n)}0, \tag{4.28}$$

that is,

$$x_{n+1}-x_n=d_n\in\mathcal{D}(x_n)\quad\text{and}\quad\|d_n\|=\|\mathcal{D}(x_n)\|, \tag{4.29}$$

where

$$\emptyset\neq\mathcal{D}(x_n)=\{d\in X:\ F(x)+F'(x)d\in C\}=F'(x)^{-1}(-F(x))$$

(the nonemptiness is because that x_0 is an implementable point). Note that such $\mathcal{D}(x_n)$ is a nonempty set in Hilbert space X, d_k is unique and so is the sequence $\{x_k\}$. It remains to show that $\{x_k\}$ is the same as that generated by the extended Newton method (1.28). To see this, note first that since γ is finite, (T_{x_0},F) satisfies the weak γ-condition at x_0 on $\mathbf{B}(x_0,\frac{1}{\gamma})$ (see Proposition 4.1). Hence, by part (I) of Proposition 3.1, $\mathrm{R}(F'(x))\subseteq\mathrm{R}(T_{x_0})(=\mathrm{R}(F'(x_0)),\ \text{as } C=\{0\})$ for all $x\in\mathbf{B}(x_0,\frac{1}{\gamma})$. Together with part (II) of Proposition 3.1, one has that

$$\mathrm{R}(F'(x))=\mathrm{R}(F'(x_0))\quad\text{for all } x\in\mathbf{B}(x_0,\frac{2-\sqrt{2}}{2\gamma}). \tag{4.30}$$

Note that $x_n\in\overline{\mathbf{B}(x_0,r^*)}$ for each $n\in\mathbb{N}$, and that $\gamma\xi\leq\frac{13-3\sqrt{17}}{4}<3-2\sqrt{2}$. Then, thanks to (2.13), one has that $\|x_n-x_0\|<\frac{2-\sqrt{2}}{2\gamma}$ and so $\mathrm{R}(F'(x_n)=\mathrm{R}(F'(x_0))$ by (4.30), and hence $\mathrm{R}(F'(x_n)$ is complemented in Y (as $\mathrm{R}(F'(x_0)$ is so thanks to the given assumption for x_0). Moreover, since $\mathcal{D}(x_n)\neq\emptyset$, one knows that $-F(x_n)\in\mathrm{R}(F'(x_n))$. Recalling (4.28), (4.29), and that $C=\{0\}$, Lemma 4.2 is applicable to $F'(x_n)$ and $-F(x_n)$ in place of A and y:

$$F'(x_n)^+(-F(x_n))=\Pi_{F'(x_n)^{-1}(-F(x_n))}0=\Pi_{\mathcal{D}(x_n)}0.$$

Since $x_{n+1}-x_n\in\Pi_{\mathcal{D}(x_n)}0$ by the definition of Algorithm $\mathbf{A}(x_0)$, we have that $x_{n+1}-x_n=-F'(x_n)^+F(x_n)$. This implies that $\{x_n\}$ is generated by the extended Newton method (1.28). The proof is complete. □

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