THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
UGEB2530C Games and Strategic Thinking (2024 Spring)
Homework 3
Due date: 23:59 on May 1

Name: ______________________________ Student No.: ________________

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

______________________________ ______________________________
Signature Date

General Guidelines for Homework Submission.

- Please submit your answer as a PDF file to Gradescope through the course UGEB2530C in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer is located. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded, resulting in zero score**. Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.

1. **Ana** (player 1) and **Betty** (player 2) plays the following zero-sum game.

   First, **Ana** writes a number 1 or 2 on a card and hides it from **Betty**. Next, **Betty** writes a number 1 or 2 on a card and hides it from **Ana**. Then, **Ana** has to guess whether the sum of the two numbers is even or odd. Finally, they reveal their cards. If **Ana**'s guess is correct, **Betty** pays **Ana** an amount equal to the sum. Otherwise, **Ana** pays **Betty** an amount equal to the sum.

   (a) Draw the game tree.
   (Payoff = money won by **Ana**)

   (b) Write down all strategies of **Ana** and **Betty** in the game's normal form.

   (c) Find a maximin strategy of **Ana**, a minimax strategy of **Betty** and **Ana**'s corresponding payoff.

2. The summer term is starting soon. Joseph and Patrick plan to move to one of the four student hostels at New Asia College from their homes by a cargo van. If they each hire a cargo van, the costs for Joseph and Patrick are \$52 and \$94 respectively. If they hire a cargo van together, the price will be \$104.

   By converting it to a saving game (which is superadditive) and considering Shapley's values, find a suitable way for Joseph and Patrick to divide the cost if they hire a cargo van together.

3. Three students, Aaron, Benny and Carol, each have to buy a book on Game Theory. The list price of the book is \$250. Aaron has a discount card which allows him to buy two books for \$450 and three books for \$600. Benny has a coupon which will enable him to have 20% off on the whole bill. The discount card and the coupon can be used at the same time.

   By converting it to a saving game (superadditive, compared to the list price) and considering Shapley values, find a suitable way to divide the total cost if the three students buy the books together.

4. Suppose **Albert** and **Bryan** decide to open restaurants next to each other. Both restaurants serve steaks with comparable quality. Due to various factors, the costs of each steak served in **Albert**'s and **Bryan**'s restaurants are \$63 and \$78 respectively. Now, each of them needs to decide whether to set the price level of a steak to *Inexpensive* (\$150) or *Expensive* (\$450). Based on the results of market research, they believe:

   - If they both choose *Inexpensive*, at each restaurant 200 steaks will be sold in one day.
   - If **Albert** chooses *Inexpensive* and **Bryan** chooses *Expensive*, then 300 steaks and 20 steaks will be sold in one day in **Albert**'s and **Bryan**'s restaurants, respectively.
   - If **Albert** chooses *Expensive* and **Bryan** chooses *Inexpensive*, then 20 steaks and 300 steaks will be sold in one day in **Albert**'s and **Bryan**'s restaurants, respectively.

- If they both choose *Expensive*, for each restaurant, 60 steaks will be sold in one day.

(a) Let **Albert** and **Bryan** be **Row** and **Col** respectively. Fill in the blanks in the following table by their estimated daily profits.

	Inexpensive	*Expensive*
Inexpensive		
Expensive		

(b) Suppose they make their decisions individually and simultaneously. What will be the expected outcome? What will be their estimated daily profits?

(c) Suppose they cooperate and make side payments if necessary. What will be the expected outcome? Based on the Shapley values, how should they arrange the side payment? Considering the side payment, what will be their estimated daily profits?

(When negotiation breaks down, we take $v(\mathbf{A})$, $v(\mathbf{B})$ to be the individual payoffs given by part (a).)

5. Let N be a coalition, namely the condition of all players. Therefore, we often call coalition N the grand coalition, $v(S)$ the value of coalition S.

- A cooperative game is called monotonic if for all $S \subset T \subseteq N$, we have

$$v(S) \leq v(T).$$

This means that the value of a coalition stays the same when the coalition grows.

- A cooperative game is additive if for all $S, T \subseteq N$ with $S \cap T = \emptyset$, we have

$$v(S \cup T) = v(S) + v(T).$$

Cooperation between players does not generate additional value if a game is additive. Hence, in that case, the situation does not incentivise collaboration.

- A cooperative game is superadditive if for all $S, T \subset N$ with $S \cap T = \emptyset$, we have

$$v(S \cup T) \geq v(S) + v(T).$$

If a game is super addictive, breaking up a coalition into parts does not pay.

Consider the following cooperative game:

Coalition S	Value of Coalition $v(S)$
$\{1\}$	3
$\{2\}$	2
$\{3\}$	1
$\{1,2\}$	8
$\{1,3\}$	8
$\{2,3\}$	4
$\{1,2,3\}$	15

(Since we assume $v(\emptyset) = 0$, we do not need to list the value of this coalition, but we need to list the values of the remaining $2^{|N|} - 1$ coalitions in case of an $|N|$-person game.)

Verify whether this game is monotonic, additive and/or superadditive.

Extra exercises

E1. Two neighbourhood cities **A** and **B** would like to organize concerts for a world-famous singer at their city halls. They may choose to organize the concert together or separately. The table below shows the cost of and income from (in a million dollars) organizing the concerts.

	City **A**	City **B**	Total if **A** and **B** cooperate
Cost	13.2	12.7	24.3
Income	14.8	15.1	31.7

Let $v(S)$ be the total profit if the member(s) of S cooperate, which is superadditive. Use Shapley values to find how the total profit should be split between the two cities.

E2. Each TV broadcasting company, AirTV (player 1) and BoldTV (player 2), must choose to broadcast one of the two sports events, the Championship League or the National Cup, without knowing what the other company has chosen. Their payoffs are given in the following table:

AirTV	BoldTV	Payoff of AirTV	Payoff of BoldTV
Championship League	Championship League	21	19
Championship League	National Cup	36	25
National Cup	Championship League	28	26
National Cup	National Cup	23	12

(a) Given that a non-pure Nash equilibrium exists, find it and the corresponding payoffs of the two companies.

(b) Suppose the two companies cooperate based on Shapley values and make side payments if necessary. When the negotiation breaks down, we take their individual payoffs to be the payoffs given by the non-pure Nash equilibrium.

Find the strategies and payoffs of the two companies (after the side payment) in the cooperation.

E3. Three districts of Hong Kong: North District (N), Tao-Po District (P) and Sha Tin (T) District, are considering whether to rebuild a joint cycle track network. Water machines and washrooms will be Interwoven across different recharge points along the way, but most importantly, it will provide access to beautiful hikes and scenic views. The costs of the construction work are listed in the following table:

Coalition	Cost (in tens of millions of dollars)
$\{N\}$	12
$\{P\}$	6
$\{T\}$	7
$\{N, P\}$	15
$\{N, T\}$	14
$\{P, T\}$	11
$\{N, P, T\}$	20

(a) By converting it to a saving game (which is superadditive), find $\nu(S)$ for each coalition S where ν is the characteristic function.

(b) Find the Shapley values for N, P, and T.

(c) How should the three districts divide the construction costs?

E4. Johnny (**J**), Kevin (**K**) and Lewis (**L**) want to go home from a theatre by hiring a taxi. The table below shows the taxi fares if they hire a taxi together or separately.

Coalition	Taxi fare
J	51
K	55
L	63
JK	70
JL	76
KL	88
JKL	110

Let $v(S)$ be the amount that a coalition S can save by hiring a taxi together, which is superadditive.

(a) Fill in the blanks in the following table:

Coalition S	$v(S)$
$\emptyset$	
J	
K	
L	
JK	
JL	
KL	
JKL	

(b) By considering the Shapley values, find the amount that each player should pay if they all share one taxi.

E5. Three towns, A, B, and C, are considering building a joint water distribution system. The costs of the construction works are listed in the following table

Coalition	Cost (in millions of dollars)
$\{A\}$	11
$\{B\}$	7
$\{C\}$	8
$\{A, B\}$	15
$\{A, C\}$	14
$\{B, C\}$	13
$\{A, B, C\}$	20

(a) By converting it into a saving game (which is superadditive), find $v(S)$ for each coalition S.

(b) Find the Shapley values of A, B, and C.

(c) How should the three towns divide the construction cost?

E6. Three ladies, A, B, and C, have dinner together at a restaurant for their secondary school class reunion. Ladies A, B, and C have digital coupons, allowing them to enjoy a 20%, 30%, and 50% discount on the total bill, respectively. Suppose the three discounts may be used simultaneously, and each planned to spend \$125 on the meal before the discount. (A and B need to pay \$250×(1-20%)×(1-30%)=\$140 if they go to dinner together.)

By converting it to a saving game (which is superadditive) and considering Shapley's values, find a suitable way for the three ladies to divide the total cost of the dinner. Please give your answer to 2 decimal places.