**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MMAT5620:Mathematics Enhancement for Teachers 2023-2024 Term 1**
**Homework Assignment 2**
**Due Date: 24 November 2023 (Friday) before 11:59 PM**

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ____________________ Date ____________________

## General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2023R1 Mathematics Enhancement for Teachers (MMAT5620)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

  For the declaration sheet:

  Either

Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit a digital form via Gradescope. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Please be aware that you can only use a ball-point pen to write your answers for any exams.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

Please attempt to solve all the problems. Your solutions for problems 1-10 are to be submitted.

We strongly recommended that you study Extra Exercises 1-10, though you are not required to submit their solutions. Suggested solutions for all the problems will be provided.

1. Domains and ranges of standard functions can be summarized as follows:

| Function defined by an expression | Domain $\mathcal{D}$ | Range $\mathcal{R}$ |
|---|---|---|
| $y = mx + c,\ m \neq 0$ | $(-\infty, +\infty)$ | $(-\infty, +\infty)$ |
| $y = \dfrac{k}{x},\ k \neq 0$ | $\mathbb{R} \setminus \{0\}$ or $\mathbb{R} - \{0\}$ | $\mathbb{R} \setminus \{0\}$ or $\mathbb{R} - \{0\}$ |
| $y = x^{2n},\ n \in \mathbb{Z}^+$ | $(-\infty, +\infty)$ | $[0, +\infty)$ |
| $y = x^{2n-1},\ n \in \mathbb{Z}^+$ | $(-\infty, +\infty)$ | $(-\infty, +\infty)$ |
| $y = \sqrt{x}$ | $[0, +\infty)$ | $[0, +\infty)$ |
| $y = ax^2 + bx + c,\ a > 0$ | $(-\infty, +\infty)$ | $[-\Delta/4a, +\infty), \Delta = b^2 - 4ac$ |
| $y = ax^2 + bx + c,\ a < 0$ | $(-\infty, +\infty)$ | $(-\infty, -\Delta/4a], \Delta = b^2 - 4ac$ |

($\mathbb{Z}^+ = \mathbb{N}$ is the set of positive integers)

Answer each question independently. Until otherwise stated, draw your graphs by either sketching them by hand or using any computer software. Paste each graph to its corresponding question.

(a) Sketch the graphs of the following functions and find their ranges accordingly :

i. $y = \dfrac{x}{x-4}$;

ii. $y = |2 - 2x - x^2|$;

(b) Graph each of the following functions. Based on the graph, state the domain and the range and find any intercept(s).

i.

$$f(x) = \begin{cases} -e^{-x} & \text{if } x < 0; \\ -e^{x} & \text{if } x \geq 0. \end{cases}$$

ii.

$$f(x) = \begin{cases} \ln(x) & \text{if } 0 < x < 1; \\ -\ln(x) & \text{if } x \geq 1. \end{cases}$$

2. For each of the following functions, find $f^{-1}$.

(a) $f(x) = \ln(x-1)$; (b) $f(x) = 3^{x+1}$;

(c) $f(x) = \dfrac{1}{4}10^{x-1} + 5$; (d) $f(x) = \log(x+3)$.

Note that the common logarithm is the logarithm to base 10, i.e, $\log_{10} x = \log x$ and the natural logarithm is the logarithm to base $e$, i.e., $\log_e x = \ln x$.

3. For each of the following functions :

$$\text{(ii)} \quad f(x) = \log_3(-x+1); \qquad \text{(ii)} \quad f(x) = 2 + (1/4)^x .$$

   (a) Find the domain of $f$;
   (b) Graph $f$;
   (c) From the graph, determine the range and any asymptotes of $f$;
   (d) Find $f^{-1}$, the inverse of $f$;
   (e) Use $f^{-1}$ to find the range of $f$;
   (f) Graph $f^{-1}$ on the same Cartesian plane as $f$.

4. Rewrite each of the following expressions as a sum or difference of multiplies of logarithms.

$$\text{(a)} \ \log_5\left(\frac{3x^2}{(ab)^{2/3}}\right), \ x, a, b > 0; \qquad \text{(b)} \ \ln\left(\frac{3x^4\sqrt[3]{1-x}}{5(x+1)^2}\right), \ 0 < x < 1.$$

5. Rewrite each of the following expressions as a single logarithm.

$$\text{(a)} \ \frac{5}{6}\log_2(x) + \frac{2}{3}\log_2(y) - \frac{1}{2}\log_2(x) - \log_2(x);$$

$$\text{(b)} \ \frac{1}{2}(\log(x) + \log(y)) - \log(z).$$

6. Solve the following equations :
   (a) $5^{2x+1} = 6^{x-2}$;
   (b) $5^x - 5^{-x} = 6$;
   (c) $\log\sqrt[3]{x} = \sqrt{\log x}$;
   (d) $\log x = 1 - \log(x-3)$.

7. If the functions $f$ and $g$ have inverses, then it can be proved that $f \circ g$ will also have an inverse and

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

   Verify that for $f(x) = x^3$ and $g(x) = 4x + 5$.

8. Use a graphing utility to compare the long-term growth rate of the following functions as $x \to \infty$. Arrange them in ascending order according to long-term growth rates:

$$\ln x, x^x, e^{4x}, x^{100}, x^{-25}, x^{100}e^{-x}, x^{1/100}, \frac{e^{8x}}{x^{10}}.$$

9. Answer each question independently.

   (a) If

   $$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \text{to } \infty}}},$$

   then find $x$ if any.

   (b) Show that for all real values of $x$, the expression

   $$\frac{x^2 - 2x + 4}{x^2 + 2x + 4}$$

   has greatest value 3 and least value $\frac{1}{3}$.

   (c) Suppose $\alpha$ and $\beta$ are the roots of $x^2 - (x+1)p - c = 0$. Find the value of

   $$\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c}.$$

10. Answer each question independently.

    (a) Solve for $x$:

    $$\frac{1}{\log_2 x} - \frac{1}{\log_2 x - 1} < 1.$$

    (b) Solve for $x$:

    $$(15 + 4\sqrt{14})^x + (15 - 4\sqrt{14})^x = 30.$$

    (c) Solve for $x$ and $y$:

    $$\log_{10} x + \log_{10} x^{1/2} + \log_{10} x^{1/4} + \cdots = y$$

    and

    $$\frac{1 + 3 + 5 + \cdots + (2y - 1)}{4 + 7 + 10 + \cdots + (3y + 1)} = \frac{20}{7 \log_{10} x}.$$

**Extra Problem Sets**

1. Sketch the graphs of the following functions and find their ranges accordingly :

   (a) $y = \dfrac{1}{x^2+4}$.

   (b) $y = \dfrac{1}{x^2-4}$, $x \neq \pm 2$.

   (c) $y = \dfrac{x^2-4}{x-2}$.

2. Evaluate :

   (a) $\lim\limits_{x \to -\infty} 5^x$; (b) $\lim\limits_{x \to -\infty} 3^{-x}$;

   (c) $\lim\limits_{x \to \infty} \left(\dfrac{1}{5}\right)^x$; (d) $\lim\limits_{x \to \infty} e^{-x}$;

   (e) $\lim\limits_{x \to 0^+} \log_4(x)$; (f) $\lim\limits_{x \to \infty} \log_{1/2}(x)$;

   (g) $\lim\limits_{x \to 0^+} \ln x$; (h) $\lim\limits_{x \to \infty} \ln x$.

3. Verify that

$$f\left(f^{-1}(x)\right) = x \quad \text{and} \quad f^{-1}\left(f(x)\right) = x$$

   for

$$f(x) = \frac{1}{x+1} \quad \text{and} \quad f^{-1}(x) = \frac{1-x}{x}.$$

4. The given function $f$ is one-to-one. Without finding its inverse, find the domain and range of $f^{-1}$.

   (a) $f(x) = 3 + \sqrt{2x-1}$; (b) $f(x) = \ln(x-1) + 2$.

5. Find the approximate solution(s) of each of the following equations by graphing an appropriate function on a graphing utility and locating the $x-$intercept(s). Note that these equations cannot be solved by standard techniques.

   (a) $2^x = 3^{x-1} + 5^{-x}$; (b) $x^3 = e^x$.

6. A student sick with a flu virus returns to an isolated college campus of 3000 students. The number of students infected with the flu t days after the student's return is predicted by the logistic function:

$$P(t) = \frac{3000}{1 + 2999e^{-0.8905t}}.$$

(a) According to this mathematical model, how many students will be inflected with the flu after 5 days?

(b) How long will it take for one-half of the students of the student population to become inflected?

(c) How many students does the model predict will become inflected after a very long period of time?

(d) Sketch the graph of $P(t)$ by a graphing utility.

7. The average rate of change of a function $f$ over an interval $x = a$ to $x = b$ is defined to be the slope of the line segment connecting the endpoints of the curve on that interval:

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a},$$

as shown in Figure 1.

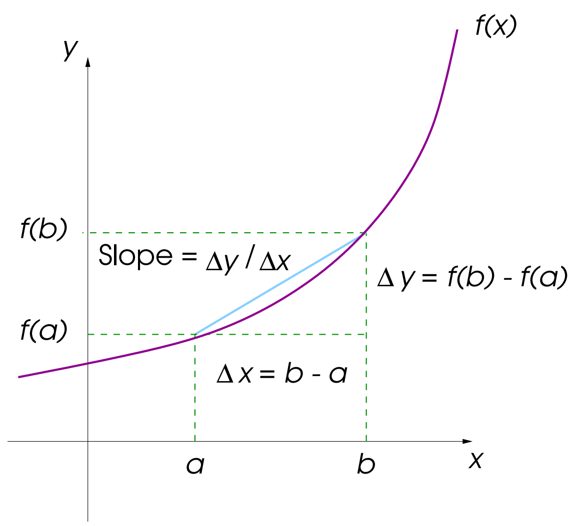


Figure 1:

The following table gives some values for the function $f(x) = x^3 + x^2 + 2x - 1$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-9$ | $-3$ | $-1$ | $3$ | $15$ | $41$ |

Answer the following questions:

(a) Find the average rate of change of $f$ from $x = -2$ to $x = 3$.

(b) Calculate the average rate of change of $f$ between each successive pair of points in the table; that is, between $x = -2$ and $x = -1$, between $x=-1$ and $x=0$, and so on. What is the average value of all these slopes?

(c) Extend the table to include the point where $x=4$ and repeat parts(a) and (b). Does the same result hold?

(d) Extend the table further to include $x=-3$. Show that the same conclusion holds.

(e) Do you think the same conclusion holds for any function and any set of points? Try to state this results as a potential theorem.

8. Answer the following questions:

(a) i. Fill in the following table:

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 |
|---|---|---|---|---|---|---|
| $f(h)=\dfrac{e^h-1}{h}$ | | | | | | |

ii. What should $\lim_{h\to 0}\dfrac{e^h-1}{h}$ be?

(b) Let $f(x)=e^x$. Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

(c) Let $f(x)=e^{2x}$. Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

(d) Let $f(x)=e^{3x}$. Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

(e) Let $f(x)=e^{nx}$ where $n\in\mathbb{N}$. Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

9. Find each of the following limits :

(a) $\lim_{x\to-\infty}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$; (b) $\lim_{x\to\infty}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$;

(c) $\lim_{x\to-\infty}\left(1+\dfrac{2e^{-x}}{e^x+e^{-x}}\right)$; (d) $\lim_{x\to\infty}\left(1+\dfrac{2e^{-x}}{e^x+e^{-x}}\right)$.

10. Find each of the following limits :

(a) $\lim_{x\to\infty}\ln\left(7x^3-x^2+1\right)$; (b) $\lim_{x\to-\infty}\ln\left(\dfrac{1}{x^2-5x}\right)$.