THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 20234-2025 Term 2
Homework Assignment 5
Due Date: 18 April, 2024 (Friday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ____________________ Date ____________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2024R2 Introductory Probability (MATH3280B)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

  For the declaration sheet:

  Either

Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit a digital form via Gradescope. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Please be aware that you can only use a ball-point pen to write your answers for any exams.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

   (a) X_1, X_2;

   (b) X_1, X_2, X_3.

2. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \quad 0 < x < 1, \ 0 < y < 2.$$

   (a) Verify that this is indeed a joint density function.

   (b) Compute the density function of X.

   (c) Find $P(X > Y)$.

   (d) Find $P(Y > 1/2 | X < 1/2)$.

   (e) Find $E[X]$.

   (f) Find $E[Y]$.

3. The joint probability density function of X and Y is given by

$$f(x,y) = e^{-(x+y)}, \ 0 \leq x < \infty, \ 0 \leq y < \infty.$$

   Find

   (a) $P(X < Y)$

   (b) $P(X < a)$

4. The joint density function of X and Y is

$$f(x,y) = \begin{cases} x + y & 0 < x < 1, \ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are X and Y independent?

   (b) Find the density function of X.

   (c) Find $P(X + Y < 1)$.

5. Suppose that A, B, C, are independent random variables, each being uniformly distributed over $(0,1)$.

   (a) What is the joint cumulative distribution function of A, B, C?

   (b) What is the probability that all of the roots of the equation $Ax^2 + Bx + C = 0$ are real?

6. If X and Y are independent and identically distributed uniform random variables on $(0,1)$, compute the joint density of

   (a) $U = X + Y$, $V = X/Y$;

(b) $U = X,\ V = X/Y$;

(c) $U = X + Y,\ V = X/(X + Y)$.

7. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1, 2, \cdots, X\}$. Call this second number Y.

   (a) Find the joint mass function of X and Y.

   (b) Find the conditional mass function of X given that $Y = 1$.

   (c) Are X and Y independent? Why?

8. The joint density function of X and Y is given by

$$f(x, y) = xe^{-x(y+1)},\ x > 0,\ y > 0.$$

   (a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.

   (b) Find the density function of $Z = XY$.

9. Let X and Y be two discrete random variables, with range

$$R_{XY} = \{(i, j) \in \mathbb{Z}^2 \big| \, i, j \geq 0,\ |i - j| \leq 1\}$$

   and joint PMF given by

$$P_{X,Y}(i, j) = \frac{1}{6 \cdot 2^{\min\{i,j\}}},\ \text{for } (i, j) \in R_{XY}.$$

   (a) Pictorially show R_{XY} in the $x - y$ plane.

   (b) Find the marginal PMFs of X and Y, i.e., $P_X(i)$ and $P_Y(j)$.

   (c) Find $P(X = Y|X < 2)$.

   (d) Find $P(1 \leq X^2 + Y^2 \leq 5)$.

   (e) Find $P(X = Y)$.

   (f) Find $E[X|Y = 2]$.

   (g) Find $\mathrm{Var}(X|Y = 2)$.

10. Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 6xy & 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show R_{XY} in the $x - y$ plane.

   (b) Find $f_X(x)$ and $f_Y(y)$.

   (c) Are X and Y independent?

   (d) Find the conditional PDF of X given $Y = y$, $f_{X|Y}(x|y)$.

   (e) Find $E[X|Y = y]$, for $0 \leq y \leq 1$.

(f) Find $\mathrm{Var}(X|Y=y)$, for $0 \leq y \leq 1$.

11. If X and Y are independent continuous positive random variables, express the density function of (a) $Z = X/Y$ and (b) $Z = XY$ in terms of the density functions of X and Y. Evaluate the density functions in the special case where X and Y are both exponential random variables.

12. Suppose that X_i, $i = 1, 2, 3$ are independent Poisson random variables with respective means λ_i, $i = 1, 2, 3$. Let $X = X_1 + X_2$ and $Y = X_2 + X_3$. The random vector X, Y is said to have a bivariate Poisson distribution. Find its joint probability mass function. That is, find $P(X = n,\ Y = m)$