**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH3280 Introductory Probability 2024-2025 Term 2**
**Homework Assignment 2**
**Due Date: 21 February, 2025 (Friday)**

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature Date

## General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2024R2 Introductory Probability (MATH3280B)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil

  For the declaration sheet:

  Either

Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF, and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit them in digital form via Gradescope. Please do not use colored paper and ensure that your written solutions are of a suitable size (easily readable). Please be aware that you can only use a ballpoint pen to write your answers for any exams.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are crucial for getting good marks on this homework. Neatness and organization are also essential.

1. If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is $i$? Compute for all values of $i$ between 2 and 12.

2. What is the probability that at least one of a pair of fair dice lands on 6, given that the sum of the dice is $i$, $i = 2, 3, \cdots, 12$?

3. Suppose we have 10 coins such that the probability of the $i$th coin showing heads when flipped is $i/10$ ($i = 1, \cdots, 10$). A coin is selected at random and flipped. If it shows heads, what is the probability that it was the fifth coin?

4. My friend has a rare genetic condition and is worried about the possibility of passing it on to her future children. There are two known mutations, $A$ and $B$. She is unsure which mutation she carries, so we'll assume that the probabilities are equal, $P(A) = P(B) = 0.5$. If she carries the $B$ mutation, there is a 1% chance of passing this mutation to her offspring. However, if she carries the $A$ mutation, the chance of transmission increases to 50%. She already has two children who do not have the condition, but she plans to have another child.

   (a) Let $K$ be the event that a child inherits the condition. What is the probability that a child inherits the condition?

   (b) Assume that the events of one child not having the condition and another child not having the condition are independent, conditional on $A$ or $B$ (i.e., $P(K_1^c \cap K_2^c|A) = P(K_1^c|A)P(K_2^c|A)$), where $K_i$ is the event that the $i$th child has the condition. What is the probability that my friend carries the $A$ mutation, $P(A|K_1^c \cap K_2^c)$?

   (c) What is the probability that the third child will not have the condition, given that the two other children do not have it?

5. $E$ and $F$ play a series of games. Each game is independently won by $E$ with probability $p$ and by $F$ with probability $1 - p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the winner of the series.

   (a) Find the probability that a total of 4 games are played.

   (b) Find the probability that $E$ is the winner of the series.

6. Let $A \subset B$. Express the following probabilities as simply as possible:

   (a) $P(A|B)$;

   (b) $P(A|B^c)$;

   (c) $P(B|A)$;

   (d) $P(B|A^c)$.

7. Mary is performing independent rolls of two dice.

   (a) What is the probability that the sum of two dice is 4?

(b) What is the probability that the { sum of two dice is 4 } appears before the outcome { sum of two dice is 8}? (*Hint:* Let $A_n$ be the event that no 4 or 8 appears on the first $n-1$ rolls and a 4 appears on the $n$th roll. Find $P\left(\cup_{n=1}^{\infty} A_n\right)$.)

8. Independent trials that result in a success with probability $q$ and a failure with probability $1-q$ are called Bernoulli trials. Let $Q_n$ denote the probability that $n$ Bernoulli trials result in an even number of successes (0 being considered an even number). Show that

$$Q_n = q(1-Q_{n-1}) + (1-q)Q_{n-1},\ n \geq 1$$

and use this formula to prove (by induction) that

$$Q_n = \frac{1+(1-2q)^n}{2}$$

9. Let $C_n$ denote the probability that no run of 3 consecutive heads appears in $n$ tosses of a fair coin. Show that

$$C_n = \frac{1}{2}C_{n-1} + \frac{1}{4}C_{n-2} + \frac{1}{8}C_{n-3}$$

$$C_0 = C_1 = C_2 = 1$$

Find $C_8$. (*Hint:* condition on the first trial)

10. For a fixed event $B$, show that the collection $P(A|B)$, defined for all events $A$, satisfies the three conditions for a probability. Conclude from this that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap C^c)P(C^c|B)$$

Then directly verify the preceding equation.

11. (a) Find $P(A|B)$

    i. if $A \cap B = \emptyset$;

    ii. if $A \subset B$;

    iii. if $A \supset B$.

    (b) Show that if $P(A|B) > P(A)$, then $P(B|A) > P(B)$.