Topics in Numerical Analysis II
Computational Inverse Problems

Lecturer: Bangti Jin (b.jin@cuhk.edu.hk)

Chinese University of Hong Kong

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Outline

Nonlinear inverse problems

$$F(x) = y,$$

with $F : X \to Y$ being nonlinear operator (possibly compact) between Hilbert spaces X and Y

(i) F is continuous

(ii) F is weakly sequentially closed, i.e., $(x_n) \subset D(F)$, weak convergence of x_n to x^* and weak convergence of $F(x_n)$ to $F(x^*)$ in Y imply $x^* \in X$ and $F(x^*) \in Y$

- electrical impedance tomography
- diffuse optical tomography
- inverse scattering
- ...

Tikhonov regularization

$$J_\alpha(x) = \|F(x) - y\|^2 + \alpha\|x\|^2$$

and approximation x_α

$$x_\alpha \in \arg\min_{x \in D(F)} J_\alpha(x)$$

- existence of a global minimizer, not unique
- stability of the approximation (on subsequence)
- consistency of the approximation as $\delta \to 0$ (subseq.)
- convergence rate

convergence rate analysis Engl-Kunisch-Neubauer Inverse Problems 1989

Let $D(F)$ be **convex**, $y^\delta \in Y$ with $\|y^\delta - y^\dagger\| \le \delta$, and $x^\dagger$ be an x_0-minimum norm solution. Moreover, the following condition holds

- **(i)** F is Frechet differentiable.
- **(ii)** $\exists\, L > 0$ s.t. $\|F'(x^\dagger) - F'(z)\| \le L\|x^\dagger - z\|$ for all $z \in D(F)$
- **(iii)** $\exists\, w \in Y$ s.t. $x^\dagger - x_0 = F'(x^\dagger)^* w$ and $L\|w\| < 1$.

Then with $\alpha \sim O(\delta)$, we have

$$\|x_\alpha^\delta - x^\dagger\| = O(\delta^{\frac{1}{2}}).$$

the same result holds for discrepancy principle

Model problem

Most inverse problems for PDEs are nonlinear in nature
model problem:

$$-\Delta u + qu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with given $q \in L^\infty(\Omega)$, $q \geq 0$, and $f \in L^2(\Omega)$

- The forward is well-posed: there exists a unique solution $u \in H_0^1(\Omega)$ (by Lax-Milgram theorem)
- Inverse problem: given $g \approx u^\delta$ in Ω, find q
- The operator $F(q) : q \mapsto u(q)$ is weakly continuous from $L^2(\Omega)$ to $L^2(\Omega)$.

Model problem

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$$-\Delta u + qu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

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The forward is well-posed: there exists a unique solution $u \in H_0^1(\Omega)$.

Lax-Milgram theorem If $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a **continuous and coercive** bilinear form on V and ℓ is a bounded linear functional on V, then the problem

$$a(u, v) = \ell(v) \quad \forall v \in V$$

has a unique solution $u \in V$.

multiply the equation with $v \in H_0^1(\Omega)$ and integration by parts

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega quv dx}_{a(u,v)} = \underbrace{\int_\Omega fv dx}_{\ell(v)}$$

continuity & coercivity of $a(\cdot,\cdot)$

$$\begin{aligned}
|a(u,v)| &\le \int_\Omega |\nabla u||\nabla v| dx + \int_\Omega q|u||v| dx \\
&\le \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le c\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}
\end{aligned}$$

$$a(u,u) \ge \|\nabla u\|_{L^2(\Omega)}^2 \ge c\|u\|_{H^1(\Omega)}^2 \quad \text{(Poincare inequality)}$$

boundedness of ℓ (linearity...)

$$|\ell(v)| \le \int_\Omega |f||v| dx \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$$

Lax-Milgram theorem $\Rightarrow \exists! \, u \in H^1(\Omega)$ (for any given q)

Inverse problem: given $g \approx u^\delta$ in Ω, find q.

How bad is the inverse problem:

$$q = \frac{f + \Delta u}{u} \qquad \text{twice differentiation in space}$$

(numerical differential is important and relatively easy !)
so the inverse problem is not that hard (in theory) ...

ill-posedness by comparing the solution for two differential potentials:

Engl, Hanke, Kunisch 1989 Inverse Problems

- $f := 16$, $u(x) = 8x(1-x)$
- $u_n = u + e_n$, with e_n, $n \geq 2$, given by

$$e_n = \begin{cases} -n^{-3/4}(2x)^{2n}, & x \leq 1/2, \\ -n^{-3/4}(2-2x)^{2n}, & x > 1/2 \end{cases}$$

- The unique sol. in $D(F)$ is $q^{\dagger} \equiv 0$ and $q_n = e_n''/(u + e_n)$
- $u_n \to u$ in $L^2(\Omega)$. but $\|q_n\|_{L^2(\Omega)} \to \infty$, i.e., $q_n \not\to q^{\dagger}$ in L^2

Thus q does not depend continuously on $F(q)$ as a mapping from $L^2(\Omega)$ to $L^2(\Omega)$

What about Tikhonov approach ?

$$J_\alpha(q) = \|F(q) - g\|^2 + \alpha\|q\|^2$$

with $F(q) : q \to u(q)$, with q belonging to $\mathcal{A}$

$$\mathcal{A} = \{q \in L^\infty(\Omega) : q \geq 0,\ q \leq c\}.$$

What about the existence of a solution to the problem ?

argument:

- $J_\alpha(q) \geq 0 \Rightarrow \exists$ a minimizing sequence $\{q^n\} \subset \mathcal{A}$

$$\lim_{n\to\infty} J_\alpha(q^n) = \inf_{q\in\mathcal{A}} J_\alpha(q)$$

- The sequence $\{q^n\}$ is uniformly bdd. in L^2
  $\Rightarrow \exists$ a subsequence $\{q^{n_k}\}$ converging weakly in $L^2(\Omega)$
- $F(q^{n_k}) \to F(q^*)$ weakly in $L^2(\Omega)$?
- weak lower semi-continuity of norm $\Rightarrow \exists$ a minimizer

weak continuity ?

weak continuity

The forward operator $F(q)$ is weakly continuous from $L^2(\Omega)$ to $L^2(\Omega)$.

Let $q^n \to q^*$ weakly in $L^2(\Omega)$

$$\int_\Omega \nabla u^n \cdot \nabla v dx + \int_\Omega q^n u^n v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

Taking $v = u^n \Rightarrow \|u^n\|_{H^1(\Omega)} \leq c$
$\Rightarrow \exists$ a subsequence $\{u^n\}$ converges weakly to u^* in $H^1(\Omega)$

$$\int_\Omega \nabla u^n \cdot \nabla v dx \to \int_\Omega \nabla u^* \cdot \nabla v dx \quad \forall v \in H_0^1(\Omega)$$

$$\underbrace{\int_\Omega q^n u^n v dx}_{\text{prod. of weakly conv. seq.}} = \int_\Omega q^n u^* v dx + \int_\Omega q^n (u^n - u^*) v dx$$

$u^*, v \in H_0^1(\Omega) \Rightarrow u^*, v \in L^4(\Omega)$ (Sobolev embedding) for $d = 2, 3$ and the embedding into $L^p(\Omega)$, $p < 4$ is compact

- $u^*, v \in L^4(\Omega) \Rightarrow u^* v \in L^2(\Omega)$ + weak conv. of q^n in $L^2(\Omega)$

$$\int_\Omega q^n u^* v dx \to \int_\Omega q^* u^* v dx$$

- $u^n \to u^*$ in $L^2(\Omega)$ + bbd of q^n

$$\left|\int_\Omega q^n(u^n - u^*) v dx\right| \le \|q^n\|_{L^\infty(\Omega)} \|u^n - u^*\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \to 0$$

- the limit u^* satisfies ($u^* = u(q^*)$)

$$\int_\Omega \nabla u^* \cdot \nabla v dx + \int_\Omega q^* u^* v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

- Every conv. subseq. of $\{u^n\}$ has a sub. seq. converging to $u(q^*)$ $\Rightarrow$ the whole sequence converges weakly to $u(q^*)$ in $H^1(\Omega)$

Is the forward operator differentiable on the set $\mathcal{A}$?

- the set $\mathcal{A}$ does not have an interior point with respect to $L^2(\Omega)$!
- (Fréchet) differentiable at $q \in \mathcal{A}$ in the sense

$$\lim_{q+h\in\mathcal{A}:\|h\|_{L^2(\Omega)}\to 0} \frac{\|F(q+h)-F(q)-F'(q)h\|}{\|h\|_{L^2(\Omega)}} = 0$$

  with $F'(q)$ being a bounded linear operator w.r.t. $L^2(\Omega)$
- Gâteaux differentiable at $q \in \mathcal{A}$ in the sense

$$\lim_{t\to 0, q+th\in\mathcal{A}} \frac{\|F(q+th)-F(q)-tF'(q)h\|}{t} = 0$$

candidate for the linearized operator

$$\int_\Omega \nabla u(q)\cdot\nabla v dx + \int_\Omega q u(q) v dx = \int_\Omega f v dx$$
$$\int_\Omega \nabla u(q+h)\cdot\nabla v dx + \int_\Omega (q+h) u(q+h) v dx = \int_\Omega f v dx$$

$\Rightarrow$ (with $\delta u = u(q+h) - u(q)$)

$$\int_\Omega \nabla \delta u\cdot\nabla v dx + \int_\Omega q\delta u\cdot\nabla v dx + \int_\Omega h u(q+h) v dx = 0$$

since $u(q+h)\approx u(q)$ for small h $\Rightarrow$ (with $\bar{u} = u'(q)h$

$$\int_\Omega \nabla \bar{u}\cdot\nabla v dx + \int_\Omega q\bar{u} v dx = -\int_\Omega h u v dx \quad \forall v\in H_0^1(\Omega)$$

sensitivity problem: $\bar{u}$ is a linear operator in h

boundedness of the operator $\bar{u}$

$$\int_\Omega \nabla\bar{u}\cdot\nabla\bar{u}dx + \int_\Omega q\nabla\bar{u}\bar{u}dx = -\int_\Omega h u\bar{u}$$

$$\leq \|h\|_{L^2(\Omega)}\|u\|_{L^4(\Omega)}\|\bar{u}\|_{L^4(\Omega)} \quad \text{(generalized Hölder inequality)}$$

$$\leq c\|h\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}\|\bar{u}\|_{H^1(\Omega)} \quad \text{(Sobolev embedding theorem)}$$

+ $\|u\|_{H^1(\Omega)} \leq c \Rightarrow$

$$\|\bar{u}\|_{H^1(\Omega)} \leq c\|h\|_{L^2(\Omega)}$$

establish the approximation property

$$\int_\Omega \nabla u(q)\cdot\nabla v dx + \int_\Omega q u(q) v dx = \int_\Omega f v dx$$
$$\int_\Omega \nabla u(q+h)\cdot\nabla v dx + \int_\Omega (q+h) u(q+h) v dx = \int_\Omega f v dx$$
$$\int_\Omega \nabla u'(q)h\cdot\nabla v dx + \int_\Omega q u'(q) h v dx = -\int_\Omega h u(q) v dx$$

$\delta u = u(q+h) - u(q) - u'(q)h \Rightarrow$

$$0 = \int_\Omega \nabla \delta u\cdot\nabla v dx + \int_\Omega q\delta u v dx + \int_\Omega h(u(q+h)-u(q)) v dx$$

letting $v = \delta u$, we deduce

$$\begin{aligned}\|\delta u\|^2_{H^1(\Omega)} &\leq c\left|\int_\Omega h(u(q+h)-u(q))\delta u dx\right| \\ &\leq c\|h\|_{L^2(\Omega)}\|u(q+h)-u(q)\|_{L^4(\Omega)}\|\delta u\|_{L^4(\Omega)} \\ &\leq c\|h\|_{L^2(\Omega)}\|u(q+h)-u(q)\|_{H^1(\Omega)}\|\delta u\|_{H^1(\Omega)}\end{aligned}$$

similar one can estimate

$$\|u(q+h)-u(q)\|_{H^1(\Omega)} \leq c\|h\|_{L^2(\Omega)} \quad \Rightarrow \quad \|\delta u\|_{H^1(\Omega)} \leq c\|h\|^2_{L^2(\Omega)}$$

$$\lim_{q+h\in\mathcal{A}:\|h\|_{L^2(\Omega)}\to 0} \frac{\|F(q+h)-F(q)-F'(q)h\|_{H^1(\Omega)}}{\|h\|_{L^2(\Omega)}} = 0$$

i.e., the forward operator is differentiable ...

convergence rates Engl-Kunish-Neubauer 1989

With $\alpha \sim \delta$, and $q^\dagger = F'(q^\dagger)^* w$ and $L\|w\| < 1$, there holds

$$\|q^\dagger - q_\alpha^\delta\| \leq c\delta^{1/2}$$

Let $A(q) : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ by

$$A(q)v = -v_{xx} + qv$$

Then $F(q) = A^{-1}(q)f$ is given by

$$F'(q)h = -A(q)^{-1}(hu)$$

What is the condition $F'(q^\dagger)^* w = q^\dagger - q_0$?

- $q^\dagger - q_0 = F'(q^\dagger)^* w = -u(q^\dagger)A(q^\dagger)^{-1} w$, with $w \in L^2(\Omega)$
- such an element exists if

$$\frac{q_0 - q^\dagger}{u(q^\dagger)} \in H^2(\Omega) \cap H_0^1(\Omega)$$

and it is given by

$$w = A(q^\dagger)\frac{q_0 - q^\dagger}{u(q^\dagger)}$$

$q_0 - q^\dagger$: should vanish near the boundary sufficiently quickly, and should be smooth in practice, one needs to discretize the problem ...

- finite difference methods
- finite element methods
- boundary element methods
- meshfree methods, particle methods
- ...

finite element methods

- versatile for general domains ...
- handle variable coefficient easily ...
- (relatively easy) theoretical justification ...

crash course on finite element methods

- $\mathcal{T}_h$: shape regular quasi-uniform triangulation of Ω, h is the mesh size, with $\Omega = \cup T$
- piecewise linear finite element space

$$V_h = \{v_h \in C_0(\bar{\Omega}) : v_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\}$$

- nodal basis $\{\varphi_i\}_{i=1}^N$ for V_h, i.e.,

$$\varphi_i(P_j) = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

finite element problem: find $u_h \in V_h$ s.t.

$$\int_\Omega \nabla u_h \cdot \nabla v_h dx + \int_\Omega q_h u_h v_h dx = \int_\Omega f v_h dx \quad \forall v_h \in V_h$$

where $q_h \in \mathcal{A}_h = \mathcal{A} \cap V_h$

- the FEM problem has a unique solution (Lax-Milgram theorem)

With the nodal basis φ_j, the FEM is given by: for $j = 1, \ldots, N$

$$\int_\Omega \nabla \sum_{i=1}^N u_i \varphi_i(x) \cdot \nabla \varphi_j(x) dx + \int_\Omega q_h(x) \sum_{i=1}^N u_i \varphi_i(x) \varphi_j(x) dx = \int_\Omega f \varphi_j(x) dx$$

resulting linear system

$$\mathbf{Au} = \mathbf{f},$$

- $\mathbf{u} = [u_1, \ldots, u_N]^t \in \mathbb{R}^N$
- $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ with

$$a_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j dx + \int_\Omega q_h \varphi_i \varphi_j dx$$

- $\mathbf{f} = [f_1, \ldots, f_N] \in \mathbb{R}^N$

All the integrals are only needed on the support, and can be evaluated analytically or using quadrature rules !!

discrete Tikhonov regularization

$$\min_{q_h \in \mathcal{A}_h} \{J_{\alpha,h}(q_h) = \tfrac{1}{2}\|F_h(q_h) - g\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|q_h\|^2_{L^2(\Omega)}\}$$

with $F_h(q_h)$ the FEM solution

- the discrete problem has a solution $q_h^* \in \mathcal{A}_h$
- what is the proper choice of h? (hard!)
- Is the discrete solution convergent ? (relatively easy!)
- It is (approximately) solvable on computers
- (the quadrature error, domain approximation error ignored)

- the finite element space V_h is dense in $H_0^1(\Omega)$ and $L^2(\Omega)$
- Ritz projection $R_h : H_0^1(\Omega) \to V_h$

$$(\nabla R_h v, \nabla v_h) = (\nabla v, \nabla v_h), \quad \forall v_h \in V_h$$

$$\lim_{h\to 0} \|v - R_h v\|_{H^1(\Omega)} = 0, \quad \forall v \in H_0^1(\Omega)$$

- Lagrange interpolation operator $I_h : C(\bar{\Omega}) \to V_h$

$$\lim_{h\to 0} \|v - I_h v\|_{C(\bar{\Omega})} = 0, \quad \forall v \in W^{1,p}(\Omega), p > d.$$

the discretization induces the error, does it affect the solution greatly ?

Does the discrete solution q_h^* approximate q^*?

Answer: Yes, there exists a subsequence $(q_h^*)_{h>0}$ converges to a solution of the continuous problem!
argument:

- constant belongs to all $\mathcal{A}_h$: $(q_h^*)_{h>0}$ is uniformly bdd in $L^2(\Omega)$
- $\exists$ a subseq. $(q_h^* \in \mathcal{A}_h)_{h>0}$ converges weakly to q^* in $L^2(\Omega)$
- $u_h(q_h^*) \to u(q^*)$ weakly in $H^1(\Omega)$
- minimizing property of q_h^* (for smooth q)

$$J_{\alpha,h}(q_h^*) \leq J_{\alpha,h}(I_h q) \to J_\alpha(q) \text{ as } h \to 0$$

$$\int_\Omega \nabla u_h(q_h^*) \cdot \nabla v_h dx + \int_\Omega q_h^* u_h(q_h^*) v_h dx = \int_\Omega f v_h dx \quad \forall v_h \in V_h$$

- with $v_h = u_h(q_h^*) \Rightarrow \|u_h(q_h^*)\|_{H^1(\Omega)} \leq c$
- convergent sub-subseq. $u_h(q_h^*)$ to u^* weakly in $H^1(\Omega)$
- for any $v \in H_0^1(\Omega)$, take $v_h = R_h v \in V_h$ + weak conv. $\Rightarrow$

$$\int_\Omega \nabla u_h(q_h^*) \cdot \nabla v_h dx \to \int_\Omega \nabla u^* \cdot \nabla v dx \quad \text{as } h \to 0$$

- splitting

$$\int_\Omega q_h^* u_h v_h dx = \int_\Omega q_h^* u^* v dx + \int_\Omega q_h^* u_h (v_h - v) dx + \int_\Omega q_h^* (u_h - u^*) v dx$$
$$\to \int_\Omega q^* u^* v dx + 0 + 0$$

by the approximation property of R_h: $v_h \to v$ in $H^1(\Omega)$

$$
\begin{aligned}
\left|\int_\Omega q_h^* u_h (v_h - v) dx\right| &\le \|q_h^*\|_{L^2(\Omega)} \|u_h\|_{L^4(\Omega)} \|v_h - v\|_{L^4(\Omega)} \\
&\le c \|q_h^*\|_{L^2(\Omega)} \|u_h\|_{H^1(\Omega)} \|v_h - v\|_{H^1(\Omega)} \to 0
\end{aligned}
$$

by the weak convergence of $u_h \to u^*$ in $H^1(\Omega)$ and compact embedding

$$
\begin{aligned}
\left|\int_\Omega q_h (u_h - u^*) v dx\right| &\le \|q_h^*\|_{L^\infty(\Omega)} \|u_h - u^*\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le c \|q_h^*\|_{L^\infty(\Omega)} \|u_h - u^*\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} \to 0
\end{aligned}
$$

+ weak lower semicontinuity of norms

$$
J_\alpha(q^*) \le \lim_{h\to 0} J_{\alpha,h}(I_h q) = J_\alpha(q) \quad \text{for any smooth } q \in \mathcal{A}
$$

a density argument shows this is true for any $q \in \mathcal{A}$

claim: $\lim_{h\to 0} J_{\alpha,h}(I_h q) = J_\alpha(q)$ for any smooth $q \in \mathcal{A}$
since q is smooth + error estimate for I_h,

$$\|q - I_h q\|_{L^\infty(\Omega)} = 0$$

meanwhile, $u_h(I_h q) \to u(q)$ in $H^1(\Omega)$

$$\int_\Omega \nabla u_h(I_h q) \cdot \nabla v_h dx + \int_\Omega I_h q u_h(I_h q) v_h dx = \int_\Omega f v_h dx \quad \forall v_h \in V_h$$

$$\int_\Omega \nabla u(q) \cdot \nabla v dx + \int_\Omega q u(q) v dx = \int_\Omega f v dx \quad \forall v \in V$$

$w_h = u(q) - u_h(I_h q)$ satisfies

$$\int_\Omega \nabla w \cdot \nabla v_h dx + \int_\Omega q w v_h dx = \int_\Omega (I_h q - q) u_h(I_h q) v_h dx, \quad \forall v_h \in V_h$$

projected gradient method

Now we need to solve the optimization problem over $q \in \mathcal{A}$

$$J_\alpha(q) = \frac{1}{2}\|F(q) - g\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}$$

gradient descent method: given q^0, solve for $k = 1, 2, \ldots$

$$q^{k+1} = P_{\mathcal{A}}(q^k - \tau J'(q^k))$$

- $\tau > 0$: a step size (Amijo's rule etc)
- $J'_\alpha(q^k) \in L^2(\Omega)$ the gradient at q^k
- $P_{\mathcal{A}}$: the projection into $\mathcal{A}$
- needs suitable stopping criterion

so one essential cost per iteration is to compute the gradient

$$J'_\alpha(q^k) = \underbrace{F'(q^k)^*}_{\text{matrix}} \underbrace{(F(q) - g)}_{\text{vector}} + \alpha q^k$$

1st approach: compute the matrix directly and multiply with vector
matrix can be computed by finite difference etc

$$[F'(q^k)]_i \approx \frac{F(q^k + he_i) - F(q^k)}{h} \quad h \text{ small}$$

where

- $[\cdot]_i$: the ith column
- e_i: the ith basis function
- h: small but not too small ...
- cost: O(n) forward solves $\sim$ very expensive (if the number of basis functions is large)

2nd approach: adjoint technique
definition of the adjoint operator

$$(F'(q^k)^*(F(q^k) - g), h)_{L^2(\Omega)} = (F(q^k) - g, F'(q^k)h)_{L^2(\Omega)} \quad \forall h \in L^2(\Omega)$$

using the definition of $F'(q^k)h$

$$\int_\Omega \nabla(F'(q^k)h)\cdot\nabla v dx + \int_\Omega q^k(F'(q^k)h) v dx = -\int_\Omega huv dx \quad \forall v \in H_0^1(\Omega) \quad (*)$$

this does not help since it does not involve $F(q^k) - g$ directly $\Rightarrow$

$$\int_\Omega \nabla v \cdot \nabla p + \int_\Omega q^k v p dx = \int_\Omega v(F(q^k) - g) dx \quad \forall v \in H_0^1(\Omega) \quad (**)$$

Taking $v = p$ in $(*)$ and $v = F'(q^k)h$ in $(**)$ $\Rightarrow$

$$(F(q^k) - g, F'(q^k)h)_{L^2(\Omega)} = -(up, h)_{L^2(\Omega)}$$

$$(F'(q)^*(F(q)-g),h)_{L^2(\Omega)} = -(up,h)_{L^2(\Omega)} \quad \forall h \in L^2(\Omega)$$

with p solving

$$\int_\Omega \nabla v \cdot \nabla p + \int_\Omega q^k vpdx = \int_\Omega v(F(q^k)-g)dx \quad \forall v \in H_0^1(\Omega)$$

$$\boxed{F'(q^k)^*(F(q)-g) = -up \quad \Rightarrow J'_\alpha(q^k) = -u(q^k)p + \alpha q^k}$$

distinct features

- Only one forward solve
- the stiffness matrix is identical with the forward problem (no extra assembling cost)

what is open for the inverse problem ?

- convergence rates of the discretization scheme ?
- convergence rates for other approximations, e.g., neural networks ?
- ...