Topics in Numerical Analysis II
Computational Inverse Problems

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Outline

Review of previous lecture

Tikhonov regularization for linear inverse problems

$$x_\alpha = \arg\min_x \|Ax - y\|^2 + \alpha\|x\|^2$$

key idea: restrict the candidate solutions to a "compact" subset

- the method is well defined: x_α exists and is unique, expressed by

$$x_\alpha = \sum_i \frac{s_i}{s_i^2 + \alpha}(y, u_i)v_i$$

- it is stable under perturbation of α and y
- it is consistent under a prior choice

$$\lim_{\delta \to 0^+} \alpha(\delta)^{-1}\delta^2 = 0 \quad \text{and} \quad \lim_{\delta \to 0^+} \alpha(\delta) = 0$$

- for $x^\dagger \in \mathrm{range}(A^*)$, it has a convergence rate $O(\delta^{\frac{1}{2}})$

Morozov's discrepancy principle: given noisy data y^δ with $\|y^\delta - y^\dagger\| = \delta$, choose α s.t.

$$\|Ax_\alpha^\delta - y^\delta\| = \delta$$

- $\alpha(\delta)$ is uniquely defined under minor conditions
- Tikhonov + Morozov is convergent
- Tikhonov + Morozov + source condition has convergence rate

(generalized) Tikhonov regularization

$$J_\alpha(x) = \|Ax - y\|^2 + \alpha\|L(x - x_0)\|^2$$

- L: differential operator
- intuition: look for a solution with small derivative $\|L(x - x_0)\| \Rightarrow$ weak compactness ...
- the penalty term often induces a (weak) compact subset in Sobolev spaces

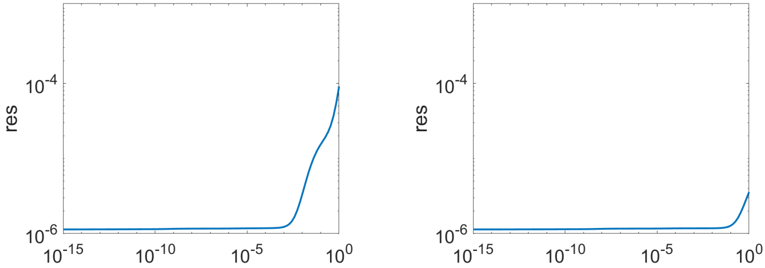
The implementation of Tikhonov regularization is as easy as for the standard case. If $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, the operator $L \in \mathbb{R}^{\ell \times n}$ and the Tikhonov functional can be given as

$$J_\alpha(x) = \|Kx - z\|^2$$

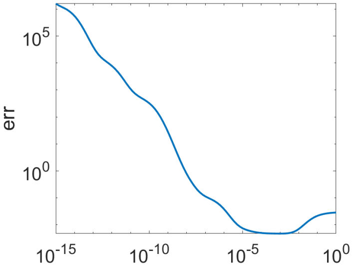
with

$$K = \begin{bmatrix} A \\ \sqrt{\alpha} L \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} y \\ \sqrt{\alpha} L x_0 \end{bmatrix}$$

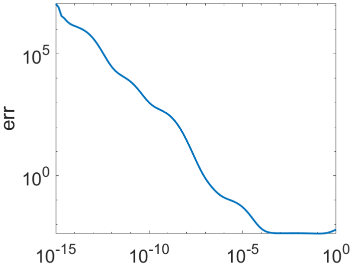
If L is properly chosen, then all singular values of K are positive and the regularized solution x_α can be computed by solving the least squares problem.



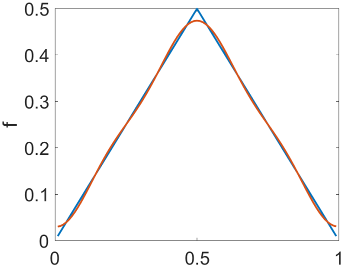
H^1 seminorm $\qquad$ H^2 seminorm



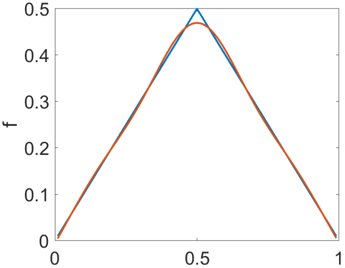
H^1 seminorm



H^2 seminorm



H^1 seminorm



H^2 seminorm

Nonlinear inverse problems

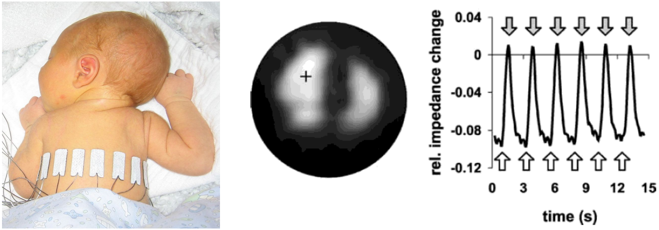
$$F(x) = y,$$

with $F : X \to Y$ being nonlinear operator (possibly compact) between Hilbert spaces X and Y

(i) F is continuous

(ii) F is weakly sequentially closed, i.e., $(x_n) \subset D(F)$, weak convergence of x_n to $x^* \in X$ and weak convergence of $F(x_n)$ to y^* in Y imply $y^* = F(x^*) \in Y$

model nonlinear inverse problem: electrical impedance tomography

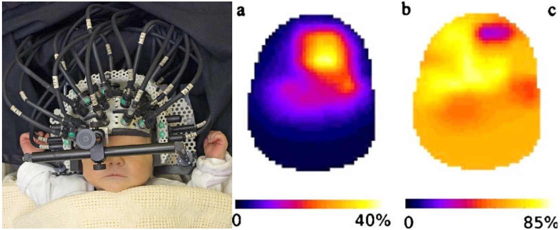


mathematical model

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = 0, & \text{in } \Omega \\ \sigma \partial_\nu u = f, & \text{on } \partial\Omega \end{cases}$$

with many pairs of boundary data (Neumann to Dirichlet map)
goal: to recover the conductivity σ from DtN map

model nonlinear inverse problem: optical tomography
blood volume / oxygen saturation of infant



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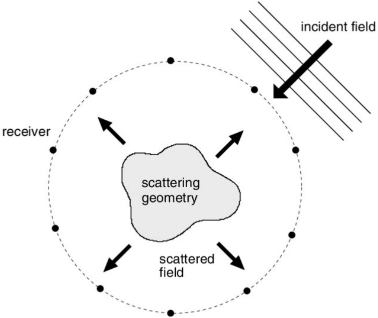
mathematical model (radiative transfer equation)

$$v \cdot \nabla_x u + \sigma(x) u = \int_{\mathbb{S}} k(x, v, v') u(x, v') \mathrm{d}v',$$

goal: recover scattering kernel k and absorption coefficient σ

inverse scattering problems



with incident wave $u^i = e^{\mathrm{i}kx\cdot \mathrm{d}}$, $u = u^i + u^s$ satisfies,

$$\begin{cases} -\Delta u - k^2 n(x)^2 u = 0, & \text{in } \mathbb{R}^d, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} (\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s) = 0 & r = |x|. \end{cases}$$

applications: remote sensing, radar

Tikhonov regularization

$$J_\alpha(x) = \|F(x) - y\|^2 + \alpha\|x\|^2$$

and approximation x_α

$$x_\alpha \in \arg\min_{x \in D(F)} J_\alpha(x)$$

x_0-minimum norm solution $x^\dagger$

$$\|F(x^\dagger) - y^\dagger\| = \min\{\|F(x) - y^\dagger\| : x \in D(F)\}$$

and the set is denoted by S, and

$$\|x^\dagger - x_0\| = \min_{x \in S} \|x - x_0\|$$

- generally the existence and uniqueness of $x^\dagger$ is nontrivial
- existence of x_0-minimum norm solution
- the choice of x_0 is crucial in practice

existence of a minimizer (by compactness argument)

- $J_\alpha(x) \geq 0 \Rightarrow \exists$ a minimizing sequence $(x^n) \subset D(F)$

$$\lim_{n\to\infty} J_\alpha(x^n) = \inf_{x\in D(F)} J_\alpha(x)$$

- The sequence (x^n) is uniformly bdd. in X
  The sequence $(F(x^n))$ is uniformly bdd. in Y
- $\exists$ a subsequence (x^{n_k}) converging to $x^* \in D(F)$ weakly in X, and $(F(x^{n_k}))$ converges weakly to $F(x^*)$ (by weak seq. closedness of F)
- weak lower semi-continuity of norm $\Rightarrow \exists$ a minimizer

the whole argument holds only on a subsequence stability of the x_α

Let $\alpha > 0$, and (y^k) and (x^k) be seq. with $y^k \to y^\delta$ and x^k be a minimizer of J_α (for y^k). Then there exists a convergent subseq. of (x^k) and the limit of every convergent subseq. is a minimizer of J_α.

The preceding argument implies that x^* is a minimizer of J_α, and

$$\lim_{k\to\infty} \|F(x^k) - y^k\|^2 + \alpha\|x^k - x_0\|^2 = \|F(x^*) - y^\delta\|^2 + \alpha\|x^* - x_0\|^2.$$

If $x^k \not\to x^*$, then

$$c := \limsup \|x^k - x_0\| > \|x^* - x_0\|$$

and there exists a subseq. of (x^k) s.t. $x^{k_j} \rightharpoonup x^*$, $F(x^{k_j}) \rightharpoonup F(x^*)$ and $\lim_{k\to\infty} \|x^{k_j} - x_0\| = c$. Thus we obtain

$$\lim_{j\to\infty} \|F(x^{k_j}) - y^{k_j}\|^2 = \|F(x^*) - y^\delta\|^2 + \alpha(\|x^* - x_0\|^2 - \lim_{j\to\infty} \|x^{k_j} - x_0\|^2)$$
$$= \|F(x^*) - y^\delta\|^2 + \alpha(\|x^* - x_0\|^2 - c^2) < \|F(x^*) - y^\delta\|^2$$

contradicts the weak lower semicontinuity. This shows $x^k \to x^*$.

convergence:

$(y^\delta)_{\delta>0}$ be seq. with $\delta = \|y^\delta - y^\dagger\|$ and $x^\delta_{\alpha(\delta)}$ be a minimizer of $J_{\alpha(\delta)}$ (for y^δ). If

$$\lim_{\delta\to 0^+} \alpha(\delta) = \lim_{\delta\to 0^+} \alpha(\delta)^{-1}\delta^2 = 0$$

then there exists a convergent subseq. of $(x^\delta_{\alpha(\delta)})$ and the limit of every convergent subseq. is an x_0-minimum norm solution.

convergence rate analysis

Let $D(F)$ be **convex**, $y^\delta \in Y$ with $\|y^\delta - y^\dagger\| \leq \delta$, and $x^\dagger$ be an x_0-minimum norm solution. Moreover, the following condition holds

(i) F is Frechet differentiable.

(ii) $\exists\, L > 0$ s.t. $\|F'(x^\dagger) - F'(z)\| \leq L\|x^\dagger - z\|$ for all $z \in D(F)$

(iii) $\exists\, w \in Y$ s.t. $x^\dagger - x_0 = F'(x^\dagger)^* w$ and $L\|w\| < 1$.

Then with $\alpha \sim O(\delta)$, we have

$$\|x_\alpha^\delta - x^\dagger\| = O(\delta^{\frac{1}{2}}).$$

- minimizing property (with $x_0 = 0$)

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta\|^2 \le \|F(x^\dagger) - y^\delta\|^2 + \alpha\|x^\dagger\|^2$$

- completing squares

$$\begin{aligned}
&\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \\
\le\ &\|F(x^\dagger) - y^\delta\|^2 - 2\alpha(x^\dagger, x_\alpha^\delta - x^\dagger) \\
=\ &\|F(x^\dagger) - y^\delta\|^2 - 2\alpha(F'(x^\dagger)^* w, x_\alpha^\delta - x^\dagger) \\
=\ &\|F(x^\dagger) - y^\delta\|^2 - 2\alpha(w, F'(x^\dagger)(x_\alpha^\delta - x^\dagger))
\end{aligned}$$

- Lipschitz continuity of the derivative:

$$\|F(x_\alpha^\delta) - F(x^\dagger) - F'(x^\dagger)(x_\alpha^\delta - x^\dagger)\| \leq \frac{L}{2}\|x_\alpha^\delta - x^\dagger\|^2,$$

  i.e.,

$$\|F'(x^\dagger)(x_\alpha^\delta - x^\dagger)\| \leq \|F(x_\alpha^\delta) - F(x^\dagger)\| + \frac{L}{2}\|x_\alpha^\delta - x^\dagger\|^2$$

- triangle inequality + Cauchy-Schwarz inequality

$$\begin{aligned}&\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha(1 - L\|w\|)\|x_\alpha^\delta - x^\dagger\|^2 \\ \leq\ &\|F(x^\dagger) - y^\delta\|^2 + \alpha\|w\|\|F(x_\alpha^\delta) - y^\delta\| + \alpha\|w\|\|y^\delta - F(x^\dagger)\|\end{aligned}$$

- Young inequality

$$\|x_\alpha^\delta - x^\dagger\| \leq c\sqrt{\delta}$$

discrepancy principle:

$$\|F(x_\alpha^\delta) - y^\delta\| = \delta$$

- under certain conditions, there exists $\alpha(\delta)$ for DP, if δ is sufficiently small
- $x_{\alpha(\delta)}^\delta \to x^\dagger$ as $\delta \to 0$, if $x^\dagger$ is unique
- under the conditions of the theorem,

$$\|x_{\alpha(\delta)}^\delta - x^\dagger\|^2 \le 4(1 - L\|w\|)^{-1}\|w\|\delta$$

Model problem

Most inverse problems for PDEs are nonlinear in nature
model problem:

$$-\Delta u + qu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with given $q \in L^\infty(\Omega)$, $q \geq 0$, and $f \in L^2(\Omega)$
The forward is well-posed: there exists a unique solution $u \in H_0^1(\Omega)$.

Lax-Milgram theorem If $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a **continuous and coercive** bilinear form on V and ℓ is a bounded linear functional on V, then the problem

$$a(u, v) = \ell(v) \quad \forall v \in V$$

has a unique solution $u \in V$.

multiply the equation with $v \in H_0^1(\Omega)$ and integration by parts

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega quv dx}_{a(u,v)} = \underbrace{\int_\Omega fv dx}_{\ell(v)}$$

continuity & coercivity of $a(\cdot,\cdot)$

$$\begin{aligned} |a(u,v)| &\leq \int_\Omega |\nabla u||\nabla v| dx + \int_\Omega q|u||v| dx \\ &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq c\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ a(u,u) &\geq \|\nabla u\|_{L^2(\Omega)}^2 \geq c\|u\|_{H^1(\Omega)}^2 \quad \text{(Poincare inequality)} \end{aligned}$$

boundedness of ℓ (linearity...)

$$|\ell(v)| \leq \int_\Omega |f||v| dx \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$$

Lax-Milgram theorem $\Rightarrow \exists!\, u \in H^1(\Omega)$ (for any given q)

Inverse problem: given $g \approx u^\delta$ in Ω, find q.

How bad is the inverse problem:

$$q = \frac{f + \Delta u}{u} \qquad \text{twice differentiation in space}$$

(numerical differential is important and relatively easy !)
so the inverse problem is not that hard (in theory) ...

ill-posedness by comparing the solution for two differential potentials:

Engl, Hanke, Kunisch 1989 Inverse Problems

- $f := 16$, $u(x) = 8x(1-x)$
- $u_n = u + e_n$, with e_n, $n \geq 2$, given by

$$e_n = \begin{cases} -n^{-3/4}(2x)^{2n}, & x \leq 1/2, \\ -n^{-3/4}(2-2x)^{2n}, & x > 1/2 \end{cases}$$

- The unique sol. in $D(F)$ is $q^\dagger \equiv 0$ and $q_n = e_n''/(u+e_n)$
- $u_n \to u$ in $L^2(\Omega)$. but $\|q_n\|_{L^2(\Omega)} \to \infty$, i.e., $q_n \not\to q^\dagger$ in L^2

Thus q does not depend continuously on $F(q)$ as a mapping from $L^2(\Omega)$ to $L^2(\Omega)$

What about Tikhonov approach ?

$$J_\alpha(q) = \|F(q) - g\|^2 + \alpha\|q\|^2$$

with $F(q) : q \to u(q)$, with q belonging to $\mathcal{A}$

$$\mathcal{A} = \{q \in L^\infty(\Omega) : q \geq 0,\ q \leq c\}.$$

What about the existence of a solution to the problem ?

argument:

- $J_\alpha(q) \geq 0 \Rightarrow \exists$ a minimizing sequence $\{q^n\} \subset \mathcal{A}$

$$\lim_{n\to\infty} J_\alpha(q^n) = \inf_{q\in\mathcal{A}} J_\alpha(q)$$

- The sequence $\{q^n\}$ is uniformly bdd. in L^2
  $\Rightarrow \exists$ a subsequence $\{q^{n_k}\}$ converging weakly in $L^2(\Omega)$
- $F(q^{n_k}) \to F(q^*)$ weakly in $L^2(\Omega)$?
- weak lower semi-continuity of norm $\Rightarrow \exists$ a minimizer

weak continuity ?

weak continuity

The forward operator $F(q)$ is weakly continuous from $L^2(\Omega)$ to $L^2(\Omega)$.

Let $q^n \to q^*$ weakly in $L^2(\Omega)$

$$\int_\Omega \nabla u^n \cdot \nabla v dx + \int_\Omega q^n u^n v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

Taking $v = u^n \Rightarrow \|u^n\|_{H^1(\Omega)} \leq c$

$\Rightarrow \exists$ a subsequence $\{u^n\}$ converges weakly to u^* in $H^1(\Omega)$

$$\int_\Omega \nabla u^n \cdot \nabla v dx \to \int_\Omega \nabla u^* \cdot \nabla v dx \quad \forall v \in H_0^1(\Omega)$$

$$\underbrace{\int_\Omega q^n u^n v dx}_{\text{prod. of weakly conv. seq.}} = \int_\Omega q^n u^* v dx + \int_\Omega q^n (u^n - u^*) v dx$$

$u^*, v \in H_0^1(\Omega) \Rightarrow u^*, v \in L^4(\Omega)$ (Sobolev embedding) for $d = 2, 3$ and the embedding into $L^p(\Omega)$, $p < 4$ is compact

- $u^*, v \in L^4(\Omega) \Rightarrow u^* v \in L^2(\Omega)$ + weak conv. of q^n in $L^2(\Omega)$

$$\int_\Omega q^n u^* v dx \to \int_\Omega q^* u^* v dx$$

- $u^n \to u^*$ in $L^2(\Omega)$ + bbd of q^n

$$|\int_\Omega q^n(u^n - u^*) v dx| \le \|q^n\|_{L^\infty(\Omega)} \|u^n - u^*\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \to 0$$

- the limit u^* satisfies ($u^* = u(q^*)$)

$$\int_\Omega \nabla u^* \cdot \nabla v dx + \int_\Omega q^* u^* v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

- Every conv. subseq. of $\{u^n\}$ has a sub. seq. converging to $u(q^*)$
$\Rightarrow$ the whole sequence converges weakly to $u(q^*)$ in $H^1(\Omega)$