THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 8

Name: ________________________ Student ID: ___________________ Score: _________

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1. Let $f(x, y) = \sin(\pi x^2)y$.

   (a) Find the 2-nd Taylor polynomial $p(x, y)$ of f, about $(a, b) = (1, 0)$.
   Solution: The second Taylor polynomial is of the form

$$p(x,y) = f(1,0) + \left(\frac{\partial f}{\partial x}\bigg|_{(1,0)} (x-1) + \frac{\partial f}{\partial y}\bigg|_{(1,0)} (y-0) \right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\bigg|_{(1,0)} (x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\bigg|_{(1,0)} (x-1)(y-0) + \frac{\partial^2 f}{\partial y^2}\bigg|_{(1,0)} (y-0)^2 \right)$$

   The partial derivatives are

$$\frac{\partial f}{\partial x} = 2\pi x y \cos(\pi x^2), \frac{\partial f}{\partial y} = \sin(\pi x^2)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\pi y \cos(\pi x^2) - 4\pi^2 x^2 y \sin(\pi x^2), \frac{\partial^2 f}{\partial x \partial y} = 2\pi x \cos(\pi x^2), \frac{\partial^2 f}{\partial y^2} = 0$$

   At $(x, y) = (1, 0)$, we have

$$\frac{\partial f}{\partial x}\bigg|_{(1,0)} = \frac{\partial f}{\partial y}\bigg|_{(1,0)} = \frac{\partial^2 f}{\partial x^2}\bigg|_{(1,0)} = \frac{\partial^2 f}{\partial y^2}\bigg|_{(1,0)} = 0, \quad \frac{\partial^2 f}{\partial x \partial y}\bigg|_{(1,0)} = -2\pi.$$

   So, the Taylor polynomial is:

$$p(x, y) = -2\pi(x - 1)y$$

(b) Let $(x, y) = (1.1, -0.01)$. Approximate the value of $f(x, y)$ using $p(x, y)$. Then, use Taylor's Theorem to find an upper bound for the error:

$$|f(x, y) - p(x, y)|.$$

Solution: The error of the approximation is

$$\left| \frac{1}{3!} \left(\left.\frac{\partial^3 f}{\partial x^3}\right|_P (0.1)^3 + 3 \left.\frac{\partial^3 f}{\partial x^2 \partial y}\right|_P (0.1)^2(-0.01) \right.\right.$$

$$\left.\left. +3 \left.\frac{\partial^3 f}{\partial x \partial y^2}\right|_P (0.1)(-0.01)^2 + \left.\frac{\partial^3 f}{\partial y^3}\right|_P (-0.01)^3 \right) \right|,$$

where $P = (1 + 0.1c, -0.01c)$, for some $c \in (0, 1)$.
Note that

$$\begin{aligned} \frac{\partial^3 f}{\partial x^3} &= -12\pi^2 xy \sin(\pi x^2) - 8\pi^3 x^3 y \cos(\pi x^2) \\ \frac{\partial^3 f}{\partial x^2 \partial y} &= 2\pi \cos(\pi x^2) - 4\pi^2 x^2 \sin(\pi x^2) \\ \frac{\partial^3 f}{\partial x \partial y^2} &= \frac{\partial^3 f}{\partial y^3} = 0 \end{aligned}$$

When $x = 1 + 0.1c, y = -0.01c$, the error becomes

$$\frac{1}{6} \left| (-12\pi^2 xy \sin(\pi x^2) - 8\pi^3 x^3 y \cos(\pi x^2))(0.1)^3 \right.$$

$$\left. +3(2\pi \cos(\pi x^2) - 4\pi^2 x^2 \sin(\pi x^2))(0.1)^2(-0.01) \right|$$

which has an upper bound of

$$\frac{\pi}{3} \left((6\pi(1.1) + 4\pi^2(1.1)^3)(0.1)^5 + (3 + 6\pi(1.1)^2)(0.1)^4 \right)$$

as both sine and cosine functions have absolute value less than or equal to 1.

2. Find all local maxima, local minima, and saddle points of the following functions:

(a) $f(x,y) = x^2 - y^3 - 2xy + 5$

Solution:

$$\frac{\partial f}{\partial x} = 2x - 2y$$
$$\frac{\partial f}{\partial y} = -3y^2 - 2x$$

Let $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 0, 0 \rangle$, then critical points are $(0,0)$ and $(-\frac{2}{3}, -\frac{2}{3})$

Calculate the second order derivative:

$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2$$
$$\frac{\partial^2 f}{\partial y^2} = -6y$$

The Hessian matrix at $(0,0)$ is

$$H = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$$

Note that $\det H = (f_{xx}f_{yy} - (f_{xy})^2)|_{(0,0)} = -4 < 0$, hence $(0,0)$ is a saddle point.
Similarly, the Hessian matrix at $(-\frac{2}{3}, -\frac{2}{3})$ is

$$H = \begin{pmatrix} f_{xx}(-2/3,-2/3) & f_{xy}(-2/3,-2/3) \\ f_{yx}(-2/3,-2/3) & f_{yy}(-2/3,-2/3) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

and $\det H = (f_{xx}f_{yy} - (f_{xy})^2)|_{(-2/3,-2/3)} = 4 > 0$. Since $f_{xx}(-2/3) = 2 > 0$, by the Second Derivative Test

$$f(-2/3,-2/3) = (-2/3)^2 - (-2/3)^3 - 2(-2/3)(-2/3) + 5$$

is a local minimum.

(b) $f(x,y) = (x^2 + y^2)e^{-x}$

Solution:

$$\frac{\partial f}{\partial x} = 2xe^{-x} - (x^2 + y^2)e^{-x}$$
$$\frac{\partial f}{\partial y} = 2ye^{-x}$$

Let $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 0, 0 \rangle$, then critical points are $(0,0)$ and $(2,0)$ Calculate the second order derivative:

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-x} - 4xe^{-x} + (x^2 + y^2)e^{-x}$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2ye^{-x}$$

$$\frac{\partial^2 f}{\partial y^2} = 2e^{-x}$$

The Hessian matrix at $(0,0)$ is

$$H = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Note that $\det H = (f_{xx}f_{yy} - f_{xy}^2)|_{(0,0)} = 4 > 0$. Since $f_{xx}(0,0) = 2 > 0$, by the Second Derivative Test $f(0,0)$ is a local minimum.
Similarly, the Hessian matrix at $(2,0)$ is

$$H = \begin{pmatrix} f_{xx}(2,0) & f_{xy}(2,0) \\ f_{yx}(2,0) & f_{yy}(2,0) \end{pmatrix} = \begin{pmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{pmatrix}$$

and $\det H = -4e^{-4} < 0$, hence $(2,0)$ is a saddle point by the Second Derivative Test.

3. **Implicit Differentiation.**

Theorem. Let $F(x, y)$ be a differentiable function such that the equation

$$F(x, y) = 0$$

defines y implicitly as a function of x. Then:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

at any point where $F_y \neq 0$.

Example. $F(x, y) = x^3 - y^2$. Then, $F(x, y) = x^3 - y^2 = 0$ defines y implicitly as a function of x near any point (x_0, y_0) where $F(x_0, y_0) = 0$ and $x_0 > 0$. More precisely, for $x, y > 0$ which satisfy $F(x, y) = 0$, we have $y = \sqrt{x^3}$. If $x > 0$, $y < 0$, and $F(x, y) = 0$, then $y = -\sqrt{x^3}$.

By the theorem, for points (x, y) which satisfy $F(x, y) = 0$, we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2}{(-2y)} \quad \text{where } y \neq 0.$$

It coincides with the $\frac{dy}{dx}$ which one obtains by differentiating $\pm\sqrt{x^3}$ directly with respect to x.

(a) Prove the theorem by differentiating both sides of:

$$F(x, y) = 0$$

with respect to x. (Note that y is implicitly a function of x, so you might want to apply the chain rule where appropriate.)

Proof: Regard y as the function of x, that is: $y = y(x)$ and differentiate both sides of $F(x, y(x)) = 0$. We have:

$$F_x + F_y \frac{dy}{dx} = 0$$

Hence at the point $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

(b) Using the theorem, find $\dfrac{dy}{dx}$ if:

i.

$$x^2y + y\sin(xy) = 0.$$

Solution:
Let

$$F(x,y) = x^2y + y\sin(xy)$$

Then

$$F_x = 2xy + y^2cos(xy)$$

$$F_y = x^2 + xycos(xy) + sin(xy)$$

Hence

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy + y^2cos(xy)}{x^2 + xycos(xy) + sin(xy)}$$

ii.

$$ye^{x^2+y^2} + \frac{x}{y} = 5$$

Solution:
Let

$$F(x,y) = ye^{x^2+y^2} + \frac{x}{y} - 5$$

Then

$$F_x = 2xye^{x^2+y^2} + \frac{1}{y}$$

$$F_y = e^{x^2+y^2} + 2y^2e^{x^2+y^2} - \frac{x}{y^2}$$

Hence

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xye^{x^2+y^2} + \frac{1}{y}}{e^{x^2+y^2} + 2y^2e^{x^2+y^2} - \frac{x}{y^2}}$$