THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 6

Name: __________________________ Student ID: _____________________ Score: __________

Show your work!

1. Evaluate each of the following limits, or show that it does not exist.

(a) $\lim_{(x,y)\to(2,-1)} \dfrac{x^3 - xy}{1-\sqrt{x}}$

Solution:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} \frac{x^3 - xy}{1-\sqrt{x}} &= \lim_{(x,y)\to(2,-1)} \frac{2^3 - 2\cdot(-1)}{1-\sqrt{2}} \\ &= \frac{10}{1-\sqrt{2}}\end{aligned}$$

(b) $\lim_{(x,y)\to(3,1)} \dfrac{x^2 - 2xy - 3y^2}{x-3y}$

Solution:

$$\begin{aligned}\lim_{(x,y)\to(3,1)} \frac{x^2 - 2xy - 3y^2}{x-3y} &= \lim_{(x,y)\to(3,1)} \frac{(x+y)(x-3y)}{x-3y} \\ &= \lim_{(x,y)\to(3,1)} (x+y) \\ &= 4\end{aligned}$$

2. Show that the following limits do not exist by computing the limits along different paths.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy^2}$

(*Hint*: Consider paths of the form $\gamma(t) = (t, mt)$, $t \in \mathbb{R}$, m a constant.)

Solution: Consider $(x(t), y(t)) = (t, mt)$ (we know that when $t \to 0$, $(x(t), y(t)) \to (0,0)$), then on this curve,

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy^2} = \lim_{t\to 0} \frac{t^3+(mt)^3}{m^2t^3}$$
$$= \frac{m^3+1}{m^2}$$

Then choose $m = 1$ and $m = 2$, the limits are 2 and $\frac{9}{4}$ respectively. Since the limits obtained on different curves do not equal to each other, the limit does not exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^3}$

(*Hint*: Consider paths of the form $\gamma(t) = (t, mt)$ and $\gamma(t) = (t^2, t), t \in \mathbb{R}$.)

Solution: First, consider $(x(t), y(t)) = (t, mt)$, then on this curve,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^3} &= \lim_{t\to 0} \frac{t^2}{m^3t^3+t^2} \\ &= 1\end{aligned}$$

Then, consider $(x(t), y(t)) = (t^2, t)$, then on this curve,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^3} &= \lim_{t\to 0} \frac{t^4}{t^4+t^3} \\ &= \lim_{t\to 0} \frac{t}{t+1} \\ &= 0\end{aligned}$$

Since the limits obtained on different curves do not equal to each other, the limit does not exist.

3. **Sandwich Theorem.** If $h(x,y) \le f(x,y) \le g(x,y)$ for all $(x,y) \ne (a,b)$ in an open neighborhood of (a,b), and:

$$\lim_{(x,y)\to(a,b)} h(x,y) = \lim_{(x,y)\to(a,b)} g(x,y) = L,$$

then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Evaluate each of the following limits:

(a) $\lim_{(x,y)\to(0,0)} xy \sin\left(\frac{1}{x+y}\right)$.

Solution: We know that $-1 \le \sin\left(\frac{1}{x+y}\right) \le 1$ for any (x,y), therefore,

$$-1 \le \sin\left(\frac{1}{x+y}\right) \le 1$$

$$-|xy| \le xy\sin\left(\frac{1}{x+y}\right) \le |xy|$$

As $\lim_{(x,y)\to(0,0)} (-|xy|) = \lim_{(x,y)\to(0,0)} |xy| = 0$, by Sandwich Theorem, we have

$$\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x+y}\right) = 0$$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$.

(*Hint*: Compare x^4+y^4 with $(x^2+y^2)^2$.)

Solution: As $(x^2+y^2)^2 = x^4+2x^2y^2+y^4 \ge x^4+y^4$,

$$\frac{x^4+y^4}{x^2+y^2} \le \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2$$

Also, since $x^4+y^4 \ge 0$ and $x^2+y^2 \ge 0$, we have

$$0 \le \frac{x^4+y^4}{x^2+y^2} \le x^2+y^2$$

As $\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$, by Sandwich Theorem, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0$$

4. Let $f(x,y) = \cos(xy^2)\sqrt{x}$.

   (a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
   Solution:

$$\frac{\partial f}{\partial x} = -\sin(xy^2)y^2\sqrt{x} + \frac{1}{2}\cos(xy^2)\frac{1}{\sqrt{x}}$$
$$\frac{\partial f}{\partial y} = -2\sin(xy^2)x^{\frac{3}{2}}y$$

   (b) Given that f is differentiable at the point $P_0 = (\pi, 1/2)$, find an equation in x, y, z whose graph is the tangent plane to the graph of f at $(x,y) = P_0$.
   Solution:

$$f(P_0) = \cos\left(\frac{1}{4}\pi\right)\sqrt{\pi} = \frac{\sqrt{2\pi}}{2}$$
$$f_x(P_0) = -\frac{1}{4}\sin\left(\frac{1}{4}\pi\right)\sqrt{\pi} + \frac{1}{2}\cos\left(\frac{\pi}{4}\right)\frac{1}{\sqrt{\pi}} = -\frac{\sqrt{2\pi}}{8} + \frac{\sqrt{2}}{4\sqrt{\pi}}$$
$$f_y(P_0) = -2\sin\left(\frac{\pi}{4}\right)\frac{1}{2}\pi^{\frac{3}{2}} = -\frac{\sqrt{2}}{2}\pi^{\frac{3}{2}}$$

   Then the tangent plane is given by:

$$z = \left(-\frac{\sqrt{2\pi}}{8} + \frac{\sqrt{2}}{4\sqrt{\pi}}\right)(x-\pi) + \left(-\frac{\sqrt{2}}{2}\pi^{\frac{3}{2}}\right)\left(y - \frac{1}{2}\right) + \frac{\sqrt{2\pi}}{2}$$

   or:

$$\left(-\frac{\sqrt{2\pi}}{8} + \frac{\sqrt{2}}{4\sqrt{\pi}}\right)(x-\pi) + \left(-\frac{\sqrt{2}}{2}\pi^{\frac{3}{2}}\right)\left(y - \frac{1}{2}\right) - \left(z - \frac{\sqrt{2\pi}}{2}\right) = 0$$