THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 4

Name: ________________________ Student ID: ___________________ Score: __________

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1. Let $\vec{v} = \langle 1, -2, 4 \rangle$, $\vec{w} = \langle 0, 5, -7 \rangle$. Find:
   (a) The unit vectors associated with $\vec{v}$ and $\vec{w}$.
   (b) The angle θ $(0 \le \theta \le \pi)$ between $\vec{v}$ and $\vec{w}$.
   (c) The vector $\mathrm{Proj}_{\vec{w}}\vec{v}$.

Solution:

(a)

$$|\vec{v}| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$$
$$|\vec{w}| = \sqrt{0^2 + (5)^2 + (-7)^2} = \sqrt{74}$$

The unit vector associated with $\vec{v}$ is $\dfrac{1}{\sqrt{21}}\langle 1, -2, 4 \rangle$.

The unit vector associated with $\vec{w}$ is $\dfrac{1}{\sqrt{74}}\langle 0, 5, -7 \rangle$.

(b)

$$\vec{v} \cdot \vec{w} = -38$$
$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} \approx -0.964$$

Hence,

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right) \approx 2.87$$

(c)

$$\mathrm{Proj}_{\vec{w}}\vec{v} = \left(v \cdot \frac{\vec{w}}{|\vec{w}|}\right)\frac{\vec{w}}{|\vec{w}|} = \left\langle 0, -\frac{95}{37}, \frac{133}{37} \right\rangle$$

2. Show that, in general, for nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, the vector $\vec{v}_\perp := \vec{v} - \mathrm{Proj}_{\vec{w}}\vec{v}$ is perpendicular to $\vec{w}$.

Proof.

$$\begin{aligned}
\vec{v}_\perp \cdot \vec{w} &= (\vec{v} - \mathrm{Proj}_{\vec{w}}\vec{v}) \cdot \vec{w} \\
&= \vec{v} \cdot \vec{w} - \mathrm{Proj}_{\vec{w}}\vec{v} \cdot \vec{w} \\
&= \vec{v} \cdot \vec{w} - \left(\left(v \cdot \frac{\vec{w}}{|\vec{w}|} \right) \frac{\vec{w}}{|\vec{w}|} \right) \cdot \vec{w} \\
&= \vec{v} \cdot \vec{w} - (\vec{v} \cdot \vec{w}) \left(\frac{\vec{w} \cdot \vec{w}}{|\vec{w}|^2} \right) \\
&= \vec{v} \cdot \vec{w} - (\vec{v} \cdot \vec{w}) \left(\frac{|\vec{w}|^2}{|\vec{w}|^2} \right) \\
&= \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} = 0
\end{aligned}$$

Hence, $\vec{v}_\perp$ is perpendicular to $\vec{w}$.

3. Let $\vec{u}_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $\vec{u}_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. Let $\vec{v} = \langle 2, 5 \rangle$.
   (a) Show that $\vec{u}_1$ and $\vec{u}_2$ are unit vectors.
   (b) Show that $\vec{u}_1$ and $\vec{u}_2$ are perpendicular to each other.
   (c) Solve for $\vec{x} \in \mathbb{R}^2$ in the matrix equation:
   $$\begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix} \vec{x} = \vec{v}.$$
   (d) Express the vectors $\text{Proj}_{\vec{u}_1}\vec{v}$ and $\text{Proj}_{\vec{u}_2}\vec{v}$ as scalar multiples of $\vec{u}_1$ and $\vec{u}_2$, respectively.
   (e) Express $\vec{v}$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$. In other words, find $s, t \in \mathbb{R}$ such that:
   $$\vec{v} = s\vec{u}_1 + t\vec{u}_2.$$

Solution:

(a)
$$|\vec{u}_1| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.$$
$$|\vec{u}_2| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.$$

(b)
$$\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = 0$$

(c) Performing Gaussian elimination on the augmented matrix:
$$\left(\begin{array}{cc|c} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{v} \\ | & | & | \end{array}\right),$$
one obtains the solution $\vec{x} = \left\langle \frac{7\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right\rangle$.

(d)
$$\text{Proj}_{\vec{u}_1}\vec{v} = \frac{\vec{v} \cdot \vec{u}_1}{|\vec{u}_1|^2}\vec{u}_1 = \frac{7\sqrt{2}}{2}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{7\sqrt{2}}{2}\vec{u}_1.$$
$$\text{Proj}_{\vec{u}_2}\vec{v} = \frac{\vec{v} \cdot \vec{u}_2}{|\vec{u}_2|^2}\vec{u}_2 = \frac{3\sqrt{2}}{2}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3\sqrt{2}}{2}\vec{u}_2.$$

(e) $\vec{v} = \frac{7\sqrt{2}}{2}\vec{u}_1 + \frac{3\sqrt{2}}{2}\vec{u}_2.$

4. Let L be the line in $\mathbb{R}^3$ parameterized by the function:

$$\vec{l} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$\vec{l} = t\langle 2, 0, -1\rangle + \langle -3, 1, 7\rangle, \quad t \in \mathbb{R}.$$

Find the (minimal) distance between the point $P = (1, 1, 5)$ and L.

Solution:

Notice that when $t = 2$, we have $\vec{l}(t) = (1, 1, 5)$. In other words, the point $P = (1, 1, 5)$ lies on the line L.

Hence, the distance between P and L is 0.

Alternatively, if we apply the distance formula:

$$d = \left| \overrightarrow{(-3, 1, 7)P} - \text{Proj}_{\langle 2,0,-1\rangle} \overrightarrow{(-3, 1, 7)P} \right|,$$

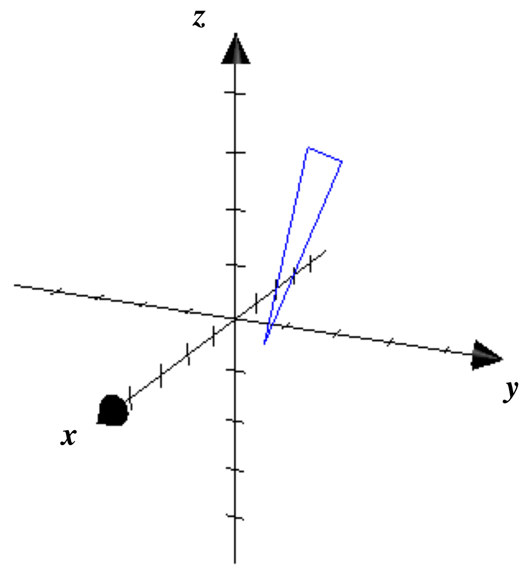
we would obtain the same answer.

5. Plot the following objects in the xyz-space.

   (a) The triangle whose vertices are the points $(1, 1, 0)$, $(0, 2, 3)$ and $(-1, 1, 3)$.

   (b) A line which is parallel to the vector $\langle 1, -1, 2\rangle$, and contains the point $(0, 0, 3)$.

(a)



(b)