THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 3

Name: ________________________ Student ID: ___________________ Score: _________

Show your work!

1. Let:

$$B = \begin{pmatrix} 0 & -2 & 3 & 4 \\ 1 & -5 & 0 & -1 \\ 0 & 1 & 6 & 0 \\ 2 & 0 & -1 & 9 \end{pmatrix}$$

Compute $\det B$ using the following two methods:

(a) Cofactor expansion along the first column.

(b) Row reduce B to an upper triangular matrix, then find $\det B$ based on the determinant of said triangular matrix.

Solution:

(a)

$$\det B = 0 \cdot \left| \begin{pmatrix} -5 & 0 & -1 \\ 1 & 6 & 0 \\ 0 & -1 & 9 \end{pmatrix} \right| + (-1) \cdot \left| \begin{pmatrix} -2 & 3 & 4 \\ 1 & 6 & 0 \\ 0 & -1 & 9 \end{pmatrix} \right|$$
$$+ 0 \cdot \left| \begin{pmatrix} -2 & 3 & 4 \\ -5 & 0 & -1 \\ 0 & -1 & 9 \end{pmatrix} \right| + (-1) \cdot 2 \cdot \left| \begin{pmatrix} -2 & 3 & 4 \\ -5 & 0 & -1 \\ 1 & 6 & 0 \end{pmatrix} \right|$$
$$= (-1) \cdot (-139) + (-2) \cdot (-135) = 409$$

(b) By row reduction, we have:

$$\begin{pmatrix} 1 & -5 & 0 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & \frac{15}{2} & 2 \\ 0 & 0 & 0 & \frac{409}{15} \end{pmatrix} = E_5 E_4 E_3 E_2 E_1 B,$$

where E_1, E_2, E_3, E_4, E_5 are the following elementary matrices:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{28}{15} & 1 \end{pmatrix}$$

Hence, we have:

$$\left| \begin{pmatrix} 1 & -5 & 0 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & \frac{15}{2} & 2 \\ 0 & 0 & 0 & \frac{409}{15} \end{pmatrix} \right| = |E_5| \cdots |E_1| \cdot |B| \, .$$

Since $|E_1| = -1$ and $|E_i| = 1$ for $i = 2, 3, 4, 5$, the determinant of B is:

$$-1 \cdot 1 \cdot (-2) \cdot \frac{15}{2} \cdot \frac{409}{15} = 409.$$

2. Let:

$$A = \begin{pmatrix} 7 & -1 & 0 \\ 0 & -2 & 3 \\ 3 & 0 & 5 \end{pmatrix}$$

(a) Find $\operatorname{adj} A$.

(b) Find A^{-1} using $\operatorname{adj} A$.

(c) (Optional) Find A^{-1} via Gaussian elimination on the augmented matrix $(A \mid I)$.

Solution:

(a)

$$\operatorname{adj} A = \begin{pmatrix} (-1)^{1+1}|M_{11}| & (-1)^{1+2}|M_{21}| & (-1)^{1+3}|M_{31}| \\ (-1)^{2+1}|M_{12}| & (-1)^{2+2}|M_{22}| & (-1)^{2+3}|M_{32}| \\ (-1)^{3+1}|M_{13}| & (-1)^{3+2}|M_{23}| & (-1)^{3+3}|M_{33}| \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -2 & 3 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 7 & 0 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 7 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 7 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 7 & -1 \\ 0 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 5 & -3 \\ 9 & 35 & -21 \\ 6 & -3 & -14 \end{pmatrix}$$

(b) Since $\det A = -79$, we have:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{1}{79} \begin{pmatrix} -10 & 5 & -3 \\ 9 & 35 & -21 \\ 6 & -3 & -14 \end{pmatrix}$$

3. Solve the linear system:

$$\begin{aligned} -3x + 7y - 4z &= 15 \\ 6x + y + 2z &= 6 \\ x - y &= -1 \end{aligned}$$

using:

(a) Cramer's rule.

(b) Gaussian elimination on the associated augmented matrix.

Solution:

(a) The linear system corresponds to $A\vec{x} = \vec{b}$, where:

$$A = \begin{pmatrix} -3 & 7 & -4 \\ 6 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 15 \\ 6 \\ -1 \end{pmatrix}.$$

Let:

$$A_1 = \begin{pmatrix} 15 & 7 & -4 \\ 6 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix}, \text{ then } |A_1| = 36;$$

$$A_2 = \begin{pmatrix} -3 & 15 & -4 \\ 6 & 6 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \text{ then } |A_2| = 72;$$

$$A_3 = \begin{pmatrix} -3 & 7 & 15 \\ 6 & 1 & 6 \\ 1 & -1 & -1 \end{pmatrix}, \text{ then } |A_3| = -36.$$

Moreover, $|A| = 36$. Hence, by Cramer's rule, the solution is:

$$x = \begin{pmatrix} \frac{|A_1|}{|A|} \\ \frac{|A_2|}{|A|} \\ \frac{|A_3|}{|A|} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(b) Row reduction on the augmented matrix:

$$\begin{pmatrix} -3 & 7 & -4 & 15 \\ 6 & 1 & 2 & 6 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 6 & 1 & 2 & 6 \\ -3 & 7 & -4 & 15 \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 7 & 2 & 12 \\ 0 & 4 & -4 & 12 \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 7 & 2 & 12 \\ 0 & 0 & -\frac{36}{7} & \frac{36}{7} \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 7 & 2 & 12 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Hence, the solution is:

$$x = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

4. Let A be any $n \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

Let B be the matrix obtained from A by switching the first and second rows:

$$B = \begin{pmatrix} a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

Without using elementary matrices or the multiplicativity of the determinant function,

(a) Show that $\det B = -\det A$.

(b) Convince yourself that if C is obtained from A by switching the i-th and $(i+1)$-st rows $(1 \leq i < n)$, then $\det C = -\det A$.

(c) Then, show that if C is a matrix obtained from A by switching *any* two rows, we have $\det C = -\det A$.

Proof:

(a) The value of $\det A$, computed using the cofactor expansion along the first row, is:

$$\det A = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} |M_{1k}|.$$

The value $\det B$, computed using the cofactor expansion along the second row, is:

$$\det B = \sum_{k=1}^{n} (-1)^{k+2} a_{1k} |M_{1k}| = (-1) \cdot \sum_{k=1}^{n} (-1)^{k+1} a_{1k} |M_{1k}|.$$

Hence, we have $\det B = -\det A$.

(b) Similarly, computing the determinants using the the cofactor expansion along the i-th and $(i+1)$-st rows, respectively, we have:

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} |M_{ik}|.$$

$$\det B = \sum_{k=1}^{n} (-1)^{i+1+k} a_{ik} |M_{ik}| = (-1) \cdot \sum_{k=1}^{n} (-1)^{i+k} a_{ik} |M_{ik}|.$$

Hence we have $\det B = -\det A$.

(c) Let $i, j \in \{1, 2, \ldots, n\}$, with $j > i$. Let C be the matrix obtained from A by switching the i-th row and the j-th row. Switching these two rows is equivalent to performing the following sequence of operations on A:

- First move the i-th row to the j-th row by switching consecutive rows as follows: Switch the i-th row with the $(i+1)$-st row, then the $(i+1)$-st with the $(i+2)$-nd, ..., and so on. This requires $j - i$ switches.
- Then, move the original j-th row, now at the $(j-1)$-st row, to the i-th row, by switching the $(j-1)$-st with the $(j-2)$-nd, and then the $(j-2)$-nd with the $(j-3)$-rd, ..., and so on. This requires $j - i - 1$ switches.

Hence, C is obtained from A by switching consecutive rows $(j-i) + (j-i-1)$ times, which implies that $\det C = (-1)^{2j-2i-1} \det A = -\det A$, since $2j - 2i - 1$ is an odd number.