THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 2

Name: ________________________ Student ID: ___________________ Score: _________

Show your work!

1. Show that if an $n \times n$ matrix A is invertible, then A^{-1} is unique. In other words, show that if there are $n \times n$ matrices B and C such that: $BA = AB = I_n$, and $CA = AC = I_n$, then $B = C$.

   Proof: Since $CA = I_n$ and $AB = I_n$, we have:

$$B = I_n B = (CA)B = C(AB) = CI_n = C.$$

2. Let A, B be $n \times n$ matrices. Let $C = AB$. Without using determinants, show that if B is non-invertible, then C is non-invertible.

   Proof: In class we have proved a theorem which says that a square matrix A is invertible if and only if $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as its unique solution.

   Since B is non-invertible, there exists a nonzero vector $\vec{x}_0 \in \mathbb{R}^n$ such that: $B\vec{x}_0 = \vec{0}$. Observe that $C\vec{x}_0 = (AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}$. In other words, the equation $C\vec{x} = \vec{0}$ has a nonzero solution $\vec{x} = \vec{x}_0$. By the same theorem just cited we conclude that C is non-invertible.

3. Let:

$$A = \begin{pmatrix} -1 & 4 & -2 \\ 0 & -3 & 3 \\ 3 & -3 & -1 \end{pmatrix}.$$

Using Gaussian elimination, row reduce the augmented matrix:

$$\left(\, A \mid I \, \right)$$

to the matrix:

$$\left(\, I \mid A^{-1} \, \right),$$

if possible. (Here, I is the 3×3 identity matrix.)

Solution:

$$(A|I) = \begin{pmatrix} -1 & 4 & -2 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 & 1 & 0 \\ 3 & -3 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

$\rightarrow$

$$\begin{pmatrix} -1 & 4 & -2 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 3 & 1 \end{pmatrix}.$$

$\rightarrow$

$$\begin{pmatrix} -1 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{7}{6} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

$\rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & \frac{5}{3} & 1 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{7}{6} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

4. Let A be an $m \times n$ matrix, and $\vec{b}$ a nonzero vector in $\mathbb{R}^m$. Suppose $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} \in \mathbb{R}^n$, must $A\vec{x} = \vec{0}$ have a unique solution?

   Conversely, if $A\vec{x} = \vec{0}$ has a unique solution, must $A\vec{x} = \vec{b}$ have a unique solution?

   Proof: Suppose $A\vec{x} = \vec{b}$ has a unique solution $\vec{v} \in \mathbb{R}^n$. Suppose $\vec{x}_0$ is a solution to $A\vec{x} = 0$, then by the linearity of matrix multiplication the vectors $\vec{x}_0 + \vec{v}$ and $\vec{v}$ are two solutions to $A\vec{x} = \vec{b}$:

$$A(\vec{x}_0 + \vec{v}) = A\vec{x}_0 + A\vec{v} = \vec{0} + \vec{b} = \vec{b}.$$

   Hence, by the uniqueness of the solution to $A\vec{x} = \vec{b}$, we have: $\vec{x}_0 + \vec{v} = \vec{v}$. In other words, $\vec{x}_0 = \vec{0}$. So $\vec{0}$ is the unique solution to $A\vec{x} = \vec{0}$.
   Conversely, suppose $A\vec{x} = \vec{0}$ has $\vec{0} \in \mathbb{R}^n$ as its unique solution. Suppose $\vec{x}_1$ and $\vec{x}_2$ are two solutions to $A\vec{x} = \vec{b}$, then $A(\vec{x}_1 - \vec{x}_2) = 0$, which implies the $\vec{x}_1 - \vec{x}_2$ is a solution to $A\vec{x} = \vec{0}$. Since by assumption $\vec{0}$ is the unique solution to $A\vec{x} = \vec{0}$, we conclude that $\vec{x}_1 = \vec{x}_2$.

5. (Optional) **LU Decomposition.**

Let:

$$A = \begin{pmatrix} 6 & -3 & 5 \\ 12 & -5 & 6 \\ -30 & 19 & -34 \end{pmatrix}$$

(a) Express A as a product $A = LU$, where L and U are triangular matrices of the form:

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

(*Hint:* Use elementary matrices to transform A to U, then find L.)

Solution: By the row reduction,

$$R = \begin{pmatrix} 6 & -3 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 7 \end{pmatrix} = E_3 E_2 E_1 A$$

where E_1, E_2, E_3 are the following elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}$$

Hence,

$$A = E_1^{-1} E_2^{-1} E_3^{-1} R = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 6 & -3 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 7 \end{pmatrix}$$

(b) Let $\vec{b} = \begin{pmatrix} 11 \\ 29 \\ -41 \end{pmatrix}$. Solve:

$$A\vec{x} = L(U\vec{x}) = \vec{b}$$

for $\vec{x} \in \mathbb{R}^3$, by performing the following steps:

i. Solve $L\vec{y} = \vec{b}$ for $\vec{y}$.

ii. Solve $U\vec{x} = \vec{y}$ for $\vec{x}$.

Remark 1: The point here is that the matrix equations (i), (ii) involve triangular matrices, so they are relatively easy to solve.

Remark 2: Once the LU decomposition is found, L and U may be used to solve $A\vec{x} = \vec{b}$ for any given $\vec{b}$, without the need to perform another Gaussian elimination on $\left(\begin{array}{c|c} A & \vec{b} \end{array} \right)$ every time a different $\vec{b}$ is given.

Solution:

i. Solve $L\vec{y} = \vec{b}$ for $\vec{y}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -5 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 29 \\ -41 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ -14 \end{pmatrix}$$

ii. Solve $U\vec{x} = \vec{y}$ for $\vec{x}$:

$$\begin{pmatrix} 6 & -3 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ -14 \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$