THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 5
Due time 6:30 pm Dec 5, 2016

1. Estimate the value of $e^{0.1}\cos(0.05)$ using the 4-th Taylor polynomial of $h(x,y)=e^x\cos y$ about $(x,y)=(0,0)$.

   Then, show that the error is no more than:

$$\frac{1}{120}\sum_{k=0}^{5}C_k^5(2)(0.1)^k(0.05)^{5-k}.$$

   Solution

   The M-th Taylor polynomial of $h(x,y)$ about (a,b) is:

$$p(x,y)=\sum_{n=0}^{M}\frac{1}{n!}\sum_{k=0}^{n}C_k^n\left.\frac{\partial^n h}{\partial x^k\partial y^{n-k}}\right|_{(x,y)=(a,b)}(x-a)^k(y-b)^{n-k}$$

   Since $h(x,y)=e^x\cos y$ is the product of two one-variable functions, it is easy to see that

$$\frac{\partial^n h}{\partial x^k\partial y^{n-k}}=\left(\frac{\partial^k}{\partial x^k}e^x\right)\left(\frac{\partial^{n-k}}{\partial y^{n-k}}\cos y\right).$$

   Hence,

$$\frac{\partial^n h}{\partial x^k\partial y^{n-k}}=\begin{cases}e^x\cos y & \text{if } n-k=0 \text{ or } 4\\ -e^x\sin y & \text{if } n-k=1\\ -e^x\cos y & \text{if } n-k=2\\ e^x\sin y & \text{if } n-k=3\end{cases}$$

   At $(x,y)=(0,0)$, we have

$$\left.\frac{\partial^n h}{\partial x^k\partial y^{n-k}}\right|_{(x,y)=(0,0)}=\begin{cases}1 & \text{if } n-k=0 \text{ or } 4\\ 0 & \text{if } n-k=1\\ -1 & \text{if } n-k=2\\ 0 & \text{if } n-k=3\end{cases}$$

   Therefore, the 4-th Taylor polynomial of f about $(x,y)=(0,0)$ is:

$$p(x,y)=1+x+\frac{1}{2}(x^2-y^2)+\frac{1}{6}(x^3-3xy^2)+\frac{1}{24}(x^4-6x^2y^2+y^4)$$

   Hence,

$$\begin{aligned}f(0.1,0.05)&\approx 1+(0.1)+\frac{1}{2}((0.1)^2-(0.05)^2)+\frac{1}{6}((0.1)^3-3(0.1)(0.05)^2)\\&\quad+\frac{1}{24}((0.1)^4-6(0.1)^2(0.05)^2+(0.05)^4)\\&\approx \boxed{1.1038}\end{aligned}$$

For the error term, we have:

$$|R_4(x,y)| = \left| \frac{1}{5!} \sum_{k=0}^{5} C_k^5 \left. \frac{\partial^5 h}{\partial x^k \partial y^{5-k}} \right|_{(x,y)=(cx,cy)} x^k y^{5-k} \right|,$$

where $c \in (0,1)$, and $(x,y) = (0.1, 0.05)$. The partial derivative $\dfrac{\partial^5 h}{\partial x^k \partial y^{5-k}}$ has the form $\pm e^x \cos(y)$ or $\pm e^x \sin(y)$. Since $|e^{cx} \cos(cy)|$ and $|e^{cx} \sin(cy)|$ are both less than or equal to $e^{0.1} < 2$, we have:

$$|R_4(0.1, 0.05)| < \frac{1}{120} \sum_{k=0}^{5} C_k^5 (2)(0.1)^k (0.05)^{5-k}.$$

2. Let $f(x,y) = \frac{1}{1+x+y}$.

   (a) Show that the 3-rd Taylor polynomial of f about $(0,0)$ is:

   $$T_3(x,y) = \sum_{n=0}^{3} (-1)^n \sum_{k=0}^{n} C_k^n x^k y^{n-k}.$$

   (b) Find a general formula for the n-th Taylor polynomial of f about $(0,0)$, where n is any positive integer.

<u>Solution</u>:

(a) 3-rd Taylor polynomial of f about $(0,0)$ is

$$T_3(x,y) = \sum_{n=0}^{3} \frac{1}{n!} \sum_{k=0}^{n} C_k^n \left. \frac{\partial^n f}{\partial x^k \partial y^{n-k}} \right|_{(x,y)=(0,0)} x^k y^{n-k}$$

We have:

$$\frac{\partial^n f}{\partial x^k \partial y^{n-k}} = (-1)^n n! \frac{1}{(1+x+y)^{n+1}}$$

$$\Rightarrow \left. \frac{\partial^n f}{\partial x^k \partial y^{n-k}} \right|_{(x,y)=(0,0)} = (-1)^n n!$$

Therefore, 3-rd Taylor polynomial of f about $(0,0)$ is:

$$T_3(x,y) = \sum_{n=0}^{3} (-1)^n \sum_{k=0}^{n} C_k^n x^k y^{n-k}$$

(b) m-th Taylor polynomial of f about $(0,0)$ is

$$T_m(x,y) = \sum_{n=0}^{m} (-1)^n \sum_{k=0}^{n} C_k^n x^k y^{n-k} = \sum_{n=0}^{m} (-1)^n (x+y)^n.$$

3. Find all local maxima, local minima, and saddle points of the following functions (Do not assume all problems must/can be solved using the Second Derivative Test.):

   (a) $f(x, y) = x^3 - y^3 - 2xy - 5$.

   (b) $f(x, y) = \dfrac{1}{1 + x^2 - y^2}$.

   (c) (Optional) $f(x, y) = \sqrt[3]{x^2 + y^2}$.

Solution:

(a)

$$f_x = 3x^2 - 2y,\ f_y = -3y^2 - 2x$$

We first find the critical points,

$$f_x = f_y = 0 \Rightarrow y = \frac{3x^2}{2},\ x = \frac{-3y^2}{2}$$

$$\Rightarrow (x, y) = (0, 0) \text{ or } \left(-\frac{2}{3}, \frac{2}{3}\right)$$

Then, we compute $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ at the critical points. We have: $f_{xx} = 6x,\ f_{xy} = f_{yx} = -2,\ f_{yy} = -6y$. Hence,

$$D\left(-\frac{2}{3}, \frac{2}{3}\right) = (-4)(-4) - (-2)^2 = 12 > 0.$$

Since, $f_{xx}(-2/3, 2/3) = -4 < 0$, by the Second Derivative Test $f(-2/3, 2/3)$ is a local maximum.

At $(0, 0)$, we have $D(0, 0) = (0)(0) - (-2)^2 = -4 < 0$, hence $(0, 0)$ is a saddle point by the Second Derivative Test.

(b)

$$f_x = \frac{-2x}{(1 + x^2 - y^2)^2},\ f_y = \frac{2y}{(1 + x^2 - y^2)^2}$$

Over the domain of f, there is only one critical point:

$$f_x = f_y = 0 \Rightarrow (x, y) = (0, 0)$$

At $(0, 0)$, $f(0, 0) = 1$, and $f(0, \varepsilon) = \dfrac{1}{1 - \varepsilon^2} > 1,\ f(\varepsilon, 0) = \dfrac{1}{1 + \varepsilon^2} < 1$, for all $\varepsilon < 1$. Hence, in every open disk centred at $(0, 0)$, there is a point at which the value of f is greater than $f(0, 0)$, and there is a point at which the value of f is smaller than $f(0, 0)$. By definition, this implies that $(0, 0)$ is a saddle point.

Alternatively, we could compute $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ at $(0, 0)$, and then apply the Second Derivative Test.

(c) For $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{1}{3}\frac{2x}{(x^2 + y^2)^{2/3}}, \quad f_y(x, y) = \frac{1}{3}\frac{2y}{(x^2 + y^2)^{2/3}}.$$

Hence, $\nabla f(x,y) \neq \langle 0,0\rangle$ for all $(x,y) \neq (0,0)$.

The function f has one critical point, namely the point $(0,0)$, where f_x and f_y are undefined.

Since the second order partial derivatives do not exist at $(0,0)$, we cannot use the Second Derivative Test.

However, observe that at all points (x,y) near (but not equal to) $(0,0)$, we have $f(x,y) = \sqrt[3]{x^2+y^2} > 0 = f(0,0)$. Hence, $f(0,0)$ is a local minimum.

4. (a) Show that:

$$\int_{-1}^{2}\int_{3}^{5}\left(x^2y^3+\frac{x}{y}\right)dy\,dx = 408+\frac{3}{2}\ln(5/3).$$

(b) Let:

$$R = \{(x,y)\in\mathbb{R}^2 \mid 0\leq x\leq \pi/2,\ -1\leq y\leq 1\}.$$

Show that:

$$\iint_R xy\cos(2x)\,dA = 0.$$

(c) Show that:

$$\int_0^2\int_x^2 y^2\cos(xy)\,dy\,dx = \frac{1}{2}(1-\cos 4).$$

(d) Show that:

$$\int_0^3\int_0^{9-y^2}\frac{ye^x}{9-x}\,dx\,dy = \frac{1}{2}(e^9-1).$$

(e) Evaluate: $\displaystyle\int_0^1\int_y^{1-y^2}\int_0^{3-x-y} y\,dz\,dx\,dy$

Solution:

(a)

$$\begin{aligned}
&\int_{-1}^{2}\int_{3}^{5}\left(x^2y^3+\frac{x}{y}\right)dy\,dx\\
&=\int_{-1}^{2}\left(\frac{1}{4}x^2y^4+x\ln y\right)\Bigg|_{y=3}^{y=5}dx\\
&=\int_{-1}^{2}\left(136x^2+(\ln 5-\ln 3)x\right)\,dx\\
&=\left(\frac{136}{3}x^3+\frac{\ln(5/3)}{2}x^2\right)\Bigg|_{x=-1}^{x=2}\\
&=408+\frac{3}{2}\ln(5/3)
\end{aligned}$$

(b)

$$\begin{aligned}
&\iint_R xy\cos(2x)\,dA \\
&= \int_0^{\pi/2}\int_{-1}^{1} xy\cos(2x)\,dy\,dx \\
&= \int_0^{\pi/2} x\cos(2x)\frac{y^2}{2}\Big|_{y=-1}^{y=1}\,dx \\
&= \int_0^{\pi/2} 0\,dx = 0
\end{aligned}$$

(c)

$$\{(x,y): 0 \le x \le 2,\ x \le y \le 2\} = \{(x,y): 0 \le y \le 2,\ 0 \le x \le y\}$$

$$\begin{aligned}
&\int_0^2\int_x^2 y^2\cos(xy)\,dy\,dx \\
&= \int_0^2\int_0^y y^2\cos(xy)\,dx\,dy \\
&= \int_0^2 y\sin(xy)\big|_{x=0}^{x=y}\,dy \\
&= \int_0^2 y\sin(y^2)\,dy \\
&= \frac{1}{2}(1-\cos 4)
\end{aligned}$$

(d)

$$\{(x,y): 0 \le y \le 3,\ 0 \le x \le 9-y^2\} = \{(x,y): 0 \le x \le 9,\ 0 \le y \le \sqrt{9-x}\}$$

$$\begin{aligned}
&\int_0^3\int_0^{9-y^2} \frac{ye^x}{9-x}\,dx\,dy \\
&= \int_0^9\int_0^{\sqrt{9-x}} \frac{e^x}{9-x}y\,dy\,dx \\
&= \int_0^9 \frac{e^x}{9-x}\left(\frac{y^2}{2}\right)\Big|_{y=0}^{y=\sqrt{9-x}}\,dx \\
&= \int_0^9 \frac{1}{2}e^x\,dx \\
&= \frac{1}{2}(e^9-1)
\end{aligned}$$

(e)

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\begin{aligned}
&\int_0^1 \int_y^{1-y^2} \int_0^{3-x-y} y\,dz\,dx\,dy \\
&= \int_0^1 \int_y^{1-y^2} yz\Big|_{z=0}^{z=3-x-y} dx\,dy \\
&= \int_0^1 \int_y^{1-y^2} \left(3y - xy - y^2\right)\,dx\,dy \\
&= \int_0^1 \left(3yx - \frac{x^2}{2}y - y^2x\right)\Bigg|_{x=y}^{x=1-y^2} dy \\
&= \int_0^1 \left(3y(1-y^2-y) - \frac{y}{2}((1-y^2)^2 - y^2) - y^2(1-y^2-y)\right)\,dy \\
&= -\frac{11}{120}
\end{aligned}
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