THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 1
Due time 6:30 pm Sep 29, 2016

1. (a) Let:

$$A = \begin{pmatrix} 4 & 10 \\ -7 & 8 \\ 6 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -7 \\ 10 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 9 & -2 \\ -3 & 1 \end{pmatrix}.$$

Verify that:

$$A(B+C) = AB + AC.$$

(b) From the definition of matrix addition and multiplication:

$$(A+B)_{ij} = A_{ij} + B_{ij}, \quad (AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj},$$

show that, for any $m \times n$ matrix A, and $n \times l$ matrices B, C, we have:

$$A(B+C) = AB + AC.$$

2. Show that, given two $m \times n$ matrices A and B, the condition $A\vec{v} = B\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$ implies that $A = B$, i.e.:

$$A_{ij} = B_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

3. (a) Let:

$$A = \begin{pmatrix} 1 & -1 & 3 & -5 \\ 2 & 0 & -1 & 3 \\ 7 & 9 & -4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 \\ 1 & 5 \\ -3 & 0 \\ 0 & -6 \end{pmatrix},$$

$$C = \begin{pmatrix} -2 & 0 & 3 & 1 \\ 5 & -7 & 0 & 4 \end{pmatrix}.$$

Verify that $(AB)C = A(BC)$.

(b) (*Optional*) Show that for any $m \times n$ matrix A, $n \times l$ matrix B, and $l \times r$ matrix C, we have:

$$A(BC) = (AB)C.$$

4. Solve the following system of linear equations by performing Gaussian elimination on the associated augmented matrix:

$$\begin{aligned} x_1 - x_2 + 5x_3 + 7x_4 &= -23 \\ 2x_1 + 4x_3 - 4x_4 &= -16 \\ 3x_2 - 2x_4 &= 0 \\ 5x_1 - x_4 &= 10 \end{aligned}$$

5. Find all solutions $\vec{x} \in \mathbb{R}^4$ to the following matrix equation:

$$\begin{pmatrix} 5 & 10 & -9 & -4 \\ 1 & 2 & 1 & 2 \\ -1 & -2 & 3 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 23 \\ -1 \\ -7 \end{pmatrix}.$$

6. For what values of $a, b, c \in \mathbb{R}$ would the following matrix equation have a unique solution $\vec{x} \in \mathbb{R}^3$?

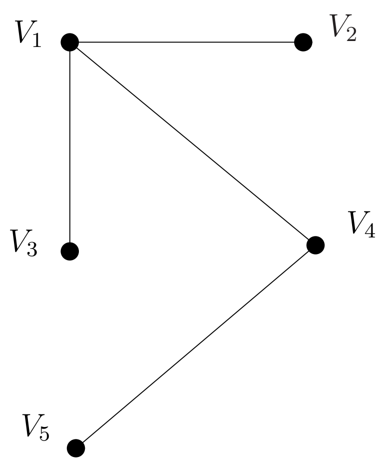
$$\begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \\ a & b & c \\ -1 & -5 & -7 \end{pmatrix} \vec{x} = \vec{0}$$

7. (Optional) An (undirected) *graph* consists of two sets of data: A set of points, called *vertices*, and a set of unordered pairs of vertices, called *edges*.

   For example, the graph with vertices $\{V_1, V_2, V_3, V_4, V_5\}$ and edges

$$\{\{V_1, V_2\}, \{V_1, V_3\}, \{V_1, V_4\}, \{V_4, V_5\}\}$$

   may be visualized as follows:



   The *adjacency matrix* of a graph with n vertices is an $n \times n$ matrix $A = (A_{ij})$ defined by:

$$A_{ij} = \begin{cases} 1 & \text{if } \{V_i, V_j\} \text{ is an edge of the graph,} \\ 0 & \text{if there is no edge connecting } V_i \text{ and } V_j. \end{cases}$$

   In the example above, the corresponding adjacency matrix is:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A *walk* in a graph is a sequence of edges linking one vertex to another. The number of edges in the sequence is called the *length* of the walk. In the example above, the sequence $\{V_1, V_4\}, \{V_4, V_5\}$ is a walk of length two from V_1 to V_5.

Prove the following theorem:

Theorem. Let $A = (A_{ij})$ be the adjacency matrix of a graph. For any integer $n \geq 1$, the number $(A^n)_{ij}$ (the ij-th entry of A^n) is equal to the number of walks of length n from V_i to V_j.