Week 9
Indefinite Integrals

Integration of Trigonometric Functions

We have seen that:

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$$

Example.

Using:

$$\int \sec^2 x \, dx = \tan x + C,$$
$$\int \csc^2 x \, dx = -\cot x + C,$$

and the identity $1 + \tan^2 x = \sec^2 x$ (which follows from the Pythagorean Theorem), we may evaluate:

- $\int \tan^2 x \, dx$
- $\int \cot^2 x \, dx$

To evaluate an integral of the form:

$$\int \sin^m x \cos^n x \, dx, \qquad n, m \in \mathbb{N},$$

it is useful to make the following substitution:

$$u = \begin{cases} \cos x, & \text{if } m \text{ is odd,} \\ \sin x, & \text{if } n \text{ is odd,} \end{cases}$$

and then apply the Pythagorean Theorem $\cos^2 x + \sin^2 x = 1$ to rewrite the original integral as:

$$\int P(u) \, du,$$

where $P(u)$ is some polynomial in u.

Example.

>

Evaluate:

$$\int \cos^5 x \sin^3 x \, dx$$

Similarly, to evaluate integrals of the form:

$$\int \tan^m x \sec^n x \, dx, \quad m, n \in \mathbb{N},$$

it is useful to make the following substitution:

$$u = \begin{cases} \sec x, & \text{if } m \text{ is odd,} \\ \tan x, & \text{if } n \text{ is even,} \end{cases}$$

and then apply the identity $1 + \tan^2 x = \sec^2 x$ to rewrite the original integral as:

$$\int P(u) \, du,$$

where $P(u)$ is some polynomial in u.

Example.

Evaluate:

- $\int \tan^3 x \sec x \, dx$.

Example.

Evaluate:

- $\int \sec^3 x \, dx$. (Hint: Consider using integration by parts.)

The following identities follow directly from the angle sum formulas of the sine and cosine functions:

$$\begin{aligned} \cos x \cos y &= \frac{1}{2}(\cos(x+y) + \cos(x-y)) \\ \cos x \sin y &= \frac{1}{2}(\sin(x+y) - \sin(x-y)) \\ \sin x \sin y &= \frac{1}{2}(\cos(x-y) - \cos(x-y)) \end{aligned}$$

They are useful for the evaluation of integrals such as:

Example.

>

$$\int \cos(3x)\sin(5x)\,dx$$

Trigonometric Substitution

When an integrand involves $\sqrt{x^2 \pm a^2}$ or $\sqrt{a^2 - x^2}$. It is sometimes useful to make the following substitution:

- $\sqrt{x^2 + a^2}$: Let $x = a\tan\theta$.
- $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$.
- $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$.

Example.

Evaluate:

- $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$
- $\displaystyle\int \frac{1}{(9+x^2)^2}\,dx$
- $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$
- $\displaystyle\int \frac{x}{8-2x-x^2}\,dx.$

Reduction Formulas

$n \in \mathbb{N}$.

- $$\underbrace{\int x^n e^x\,dx}_{I_n} = x^n e^x - n\underbrace{\int x^{n-1} e^x\,dx}_{I_{n-1}}.$$
- For $n \geq 2$,

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

- For $n \geq 2$,

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

- For $n \geq 3$,

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$$

- $$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx.$$
