Week 6
L'Hôpital's Rule
Taylor's Theorem

Theorem.

Cauchy's Mean Value Theorem. If $f, g : [a, b] \longrightarrow \mathbb{R}$ are functions which are continuous on $[a, b]$ and differentiable on (a, b), and $g(a) \neq g(b)$, then there exists $c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof.

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Exercise. Apply Rolle's Theorem to:

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

Theorem.

L'Hôpital's Rule. Let $c \in \mathbb{R}$. Let $I = (a, b)$ be an open interval containing c. Let f, g be functions which are differentiable at every point in $(a, c) \cup (c, b)$. Suppose:

- $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ are both equal to 0 or both equal to $\pm\infty$.
- $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$.
- $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists.

Then,

$$\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.}$$

Exercise.

Use l'Hôpital's rule to evaluate the following limits:

1. $\lim_{x\to 0} \frac{1 - x\cot x}{x\sin x}$

2. $\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}$

3. $\lim_{x\to +\infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$

4. $\lim_{x\to +\infty} (e^x + x)^{\frac{1}{x}}$

Definition.

Given a function f which is n times differentiable at a. The **$\mathbf{n}$-th Taylor polynomial of f (centered) at a** is:

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

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Observe that:

$$P^{(k)}(a) = f^{(k)}(a),$$

for $k = 1, 2, \ldots, n$.

Example.

The Taylor polynomials at $a = 0$ for various functions f are as follows:

$f(x)$	$P(x)$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n}$
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^k \frac{x^{2k+1}}{2k+1}$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$

Note, for example, that the 5-th and 6-th Taylor polynomials of $f(x) = \sin x$ at $x = 0$ both have degree 5. Hence, an n-th Taylor polynomial does not necessarily have degree n.

Theorem.

(**Taylor's Formula**) Let n be a positive integer, and $a \in \mathbb{R}$. Let f be a function which is $n+1$ times differentiable on an open interval I containing a. Let:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

be the n-th Taylor polynomial of f at a. Then, for any $x \in I$, we have:

$$f(x) = P_n(x) + R_n(x),$$

where the **remainder term** $R_n(x)$ is equal to:

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Note that the special case $n = 0$ is equivalent to (Lagrange's) Mean Value Theorem.
