Revision Exercise for MATH/BMED Freshmen 2018-19

- Here you will find a set of questions on various topics in **school mathematics** in which MATH/BMED freshmen are supposed to be well-versed. You are advised to attempt all questions as review in school mathematics. Do so in several sittings, each covering, say, twenty questions. Mark the amount of time you need for finishing the whole exercise (as best you could). We will send the **numerical answers** and **follow-up advice** to your CWEM account near the beginning of the semester.

Section 1: Arithmetic and algebra.

1. Classify the numbers below as rational numbers and irrational numbers. (No need to give reason.)
   (a) 5
   (b) 0.12345
   (c) $0.\dot{1} \cdot 5$
   (d) $\sqrt{3}$
   (e) $\sqrt[10]{1024}$
   (f) $\sin(30^\circ)$
   (g) $\sin(45^\circ)$
   (h) $\log_{10}(5)$
   (i) $\log_3(\sqrt{3})$
   (j) $\sqrt{\pi}$

2. Suppose $x = \dfrac{2a+1}{a-1}$ and $y = \dfrac{2a-1}{a+1}$.
   Prove that $y = \dfrac{x+B}{Cx+D}$.
   Here B, C, D are integers whose respective values you have to determine explicitly.

3. Let x be a real number greater than 1. Suppose $a = x + \dfrac{1}{x}$.
   Express each number below in terms of a.
   (a) $x^2 + \dfrac{1}{x^2}$ (b) $x - \dfrac{1}{x}$ (c) $x^2 - \dfrac{1}{x^2}$

4. Let x be a non-zero number, and m, n be integers.
   Prove that
   $$\frac{(x^{m-n})^{m+n} - 1}{(x^m)^m} \cdot x^{m^2+n^2} = Ax^{m^P} + Bx^{n^Q}.$$
   Here A, B, P, Q are integers whose respective values you have to determine explicitly.

5. Suppose $y = \dfrac{\sqrt{ax^2+b} - \sqrt{ax^2-b}}{\sqrt{ax^2+b} + \sqrt{ax^2-b}}$.
   Prove that $y + \dfrac{1}{y} = \dfrac{Ma^P x^Q}{b^R + N}$.
   Here M, N, P, Q, R are integers whose respective values you have to determine explicitly.

6. Apply mathematical induction to prove that
   $$1^2 - 2^2 + 3^2 + \cdots + (-1)^{n-1} n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}.$$

7. Apply mathematical induction to prove that
   $$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \geq \frac{n}{2}.$$

8. Solve the equation
   $$\log_{12}(x^4) = 2\log_{12}(4) + \log_{12}(3) - 1$$
   with real unknown x.

9. Suppose
   $$\begin{cases} \log_9(x) + \log_9(y+2) = t \\ \log_3(x+1) + \log_3(y) = 2t \end{cases}.$$
   Prove that x, y are related by the equation $y = Ax^B$.
   Here A, B are real constants whose respective values you have to determine explicitly.

10. Suppose $x : y = 3 : 4$. Find the respective values of:
    (a) $\dfrac{x+y}{y}$
    (b) $\dfrac{x}{y-x}$
    (c) $\dfrac{(x+y)^2}{x^2+y^2}$
    (d) $\dfrac{x^2-xy+y^2}{x^2+xy+y^2}$

11. Let x, y, z be non-zero real numbers.
    Suppose $\dfrac{x}{25} = \dfrac{y}{15} = \dfrac{\sqrt{x^2-y^2}}{z}$.
    Find the value of z.

12. Let a, b, c be real numbers. Suppose $a : b = 3 : 4$ and $a : c = 2 : 5$.
    (a) Find $a : b : c$.
    (b) Find the value of $\dfrac{ac}{a^2+b^2}$.

13. Let x, y, z be real numbers. Suppose $x : y = y : z$.
    (a) Prove that $x^2 : y^2 = x : z$.
    (b) Prove that $x : (x+y) = (x-y) : (x-z)$.

14. Let w, x, y, z be real numbers.
    Suppose $w : x = x : y = y : z$.
    Prove that $(w+x)(y+z) = (x+y)^2$.

15. Solve the equation $2^{2x+3} + 7 \cdot 2^x - 1 = 0$.

16. Solve the system of equations
    $$\begin{cases} \log_2(x) - \log_4(y) = 4 \\ \log_2(x-2y) = 5 \end{cases}.$$

17. Solve the system of equations
    $$\begin{cases} x + y - 9 = 0 \\ y = 3(x+1)^2 \end{cases}.$$

18. Solve the system of equations
    $$\begin{cases} 2x - 3y - 5 = 0 \\ 3x - 2y + 5xy = 0 \end{cases}.$$

19. Solve the system of equations

$$\left\{ \begin{array}{rcrcl} 2x^{-1} & + & y^{-1} & = & 3 \\ x^{-2} & + & y^{-2} & = & 2 \end{array} \right. .$$

20. Solve the equation $|x-3| = |x^2 - 4x + 3|$.

21. Solve the equation $\dfrac{2}{x-2} - \dfrac{1}{x-5} - \dfrac{1}{x-7} = 0$.

22. Let c be a real number.
Solve the equation $(2x-3)(x+5) = (2c-3)(c+5)$ with unknown x.

23. Let c be a real number. Suppose that the product of roots of the polynomial $cx(x+2) - (c+2)^2$ with indeterminate x is -9. Find all possible values of c.

24. Let c be a real number. Suppose that α, β are the roots of the polynomial $x^2 - cx + 10$, and $\alpha - 2\beta = 1$. Find all possible values of c.

25. Suppose α, β are the roots of the polynomial $2x^2 + 3x - 5$. For each of the pairs of the numbers below, find a quadratic polynomial whose roots are exact the pair of numbers concerned:

    (a) $\alpha - 1, \beta - 1$
    (b) $3\alpha, 3\beta$
    (c) $2\alpha + 1, 2\beta + 1$
    (d) α^{-1}, β^{-1}

26. Determine the range of (real) values of λ for which the equation

$$x^2 + 4x + 2 + \lambda(2x+1) = 0$$

has no real roots.

27. Let p be a real constant. Suppose α, β are the roots of the quadratic polynomial $x^2 + (p-2)x + p$ with indeterminate x.

    (a) Express $\alpha + \beta$ and $\alpha\beta$ in terms of p.
    (b) Suppose α, β are real numbers and $\alpha^2 + \beta^2 = 11$. Find the value(s) of p.

28. Let k be a real constant. Suppose the solution of the equation

$$x^2 + (k+2)x + 2(k-1) = 0$$

is given by $x = \alpha$ or $x = \beta$.

    (a) Prove that α and β are real and distinct.
    (b) Suppose $|\alpha - \beta| > 3$. Find the range of possible values of k.

29. Suppose p, q, k are real numbers satisfying the equations

$$\left\{ \begin{array}{rcrcrcl} p & + & q & + & k & = & 2 \\ pq & + & qk & + & kp & = & 1 \end{array} \right. .$$

    (a) Express pq in terms of k.
    (b) Find a quadratic polynomial, with coefficients in terms of k, whose roots are p and q.
    (c) Hence find all possible values of k.

30. Let $f : \mathbb{R}\backslash\{2,3\} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \frac{x-1}{(x-2)(x-3)} \qquad \text{for any —}x \in \mathbb{R}\backslash\{2,3\}.$$

Suppose (s,t) is a point on the graph of f.
Prove that $t \leq -3 - 2\sqrt{2}$ or $t \geq -3 + 2\sqrt{2}$.
(Hint: Translate the problem into something about quadratic equations.)

31. Let a, b, s, t be real numbers, with $a \neq 0$, $r \neq 0$ and $s \neq 0$. Let C be the curve given by the equation $y = ax^2 + bx + 6$, and ℓ be the line joining $(0,r), (s,t)$.
Suppose that C cuts the x-axis at $(-1,0)$ and $(5,0)$. Further suppose that ℓ is parallel to the x-axis and cuts C at $(0,r)$, (s,t). (Give a sketch of this curve on the coordinate plane before proceeding further.)

    (a) What are the respective values of a, b?
    (b) What are the respective values of r, s, t?
    (c) What is the maximum possible value of y-coordinate for a point on the curve?

32. Solve the inequality

$$(3x-1)(4x+1) < (2x-1)(6x+5).$$

33. Solve the inequality $\dfrac{x}{x-1} > 2$.

34. Solve the inequality $\log_2(\log_3(\log_4(x))) \geq 0$.

35. Solve the inequality $x(4x+3) > 14 + 2x$

36. Solve the inequality $x^2 - |x| - x < 0$.

37. Solve the inequality $\dfrac{x^2 - 6x + 5}{x^2 + 5x + 4} > 0$.

38. Solve the inequality $\left|\dfrac{3x+8}{x+1}\right| < 2$.

39. Solve the inequality $|x-3| \leq |2x-5|$.

40. Solve the system of inequalities

$$\left\{ \begin{array}{rcrcrcr} 4x & - & 3 & < & 2x & + & 7 \\ 6x & - & 12 & > & 5x & - & 15 \end{array} \right. .$$

41. Solve the system of inequalities

$$3x - 1 < x < 3(2x+5).$$

42. Solve the system of inequalities

$$\left\{ \begin{array}{rcrcrcrcr} 3 & - & 2x & - & x^2 & < & 0 & & \\ & & 7x & + & 3 & \geq & 2x & - & 7 \end{array} \right. .$$

43. Let a, b be real numbers. Suppose the solution of the inequality $2x^2 + ax + b < 0$ is $-2 < x < 3$. Find the respective values of a, b.

44. Find the coefficient of x^2 in the expansion of $\left(x - \dfrac{2}{x}\right)^6$.

45. Let n be a positive integer. Find the coefficient of x^2 in the expansion of $(1-x^2)(1+x)^n$.

46. Let a, b be real numbers. Suppose

$$(1 - ax + bx^2)^{10} = 1 - 20x + 190x^2 + \cdots$$

as polynomials (in ascending powers of x).

(a) Find the respective values of a, b.

(b) Find the coefficient of x^3 in $(1 - ax + bx^2)^{10}$.

47. Let n be a positive integer. Suppose the second and third terms in the expansion of $(x + 1)^n$, in ascending powers of x, are in the same ratio as the third and fourth term in the expansion of $(x + 1)^{n+3}$, in ascending powers of x. Find the value of n.

48. Let a, b be real numbers. Suppose that in the expansion of $(1-x)(a+bx)^8$, the coefficient of x^5 is 0. Find the ratio $a : b$.

49. Find the coefficient of x^2 in the polynomial expression

$$(1 + x)^5 + (1 + x)^6 + (1 + x)^7 + \cdots + (1 + x)^{20}.$$

(Hint: Make use of what you know about geometric progressions.)

50. Let $z = (1 - 2i)^5$.

(a) Using the Binomial Theorem, express z in the form $a + bi$ where a, b are real.

(b) Find the real part of $\frac{1}{z}$.

(c) What is the integer closest to the real part of $z + \frac{1}{z}$?

51. Find the quotient and remainder for each of the expression below by performing 'long division':

(a) $(3x^3 - 2x^2 + x - 1) \div (x - 2)$

(b) $(3x^5 - 6x^4 + 8x^3 - 11x^2 - x + 7) \div (x^2 - x + 2)$

(c) $(x^6 - 1) \div (x^2 + 1)$

52. Let a, b be real numbers. Suppose the polynomial $3x^4 + x^3 + ax^2 + 5x + b$ is divisible by $x - 1$ and $x + 2$. Find the values of a, b respectively.

53. Apply the Factor Theorem to completely factorize each of the polynomials below:

(a) $2x^3 - 7x^2 + 4x + 4$ (b) $6x^3 - 17x^2 - 4x + 3$

54. Let c be a real number, and $f(x)$ be the polynomial given by

$$f(x) = cx^4 + (c + 1)x^3 - (c + 2)x^2 - (c + 1)x + c.$$

Suppose $f(x)$ is divisible by $x - 1$.

(a) What is the value of c?

(b) Prove that $f(x)$ is divisible by $x^2 - 1$.

(c) Solve the equation $f(x) = 0$ with unknown x.

55. Let $f(x)$ be a polynomial with real coefficients. Suppose that $f(x)$ is divisible by $x - 1$. Further suppose that the remainder when $f(x)$ is divided by $x - 2$ are -7, and that the remainder when $f(x)$ is divided by $x - 3$ is -20.

When $f(x)$ is divided by $(x - 1)(x - 2)(x - 3)$, the remainder is $ax^2 + bx + c$, where a, b, c are real constants.

Find the values of a, b, c.

56. Let a, b be distinct constants, and $f(x)$ be the polynomial

$$(a - b)x^3 - (a^3 - b^3)x + a^3b - b^3a.$$

Suppose $f(x)$ is divisible by $x - a$ and $x - b$. Factorize $f(x)$ completely.

57. Let A, B, C, D be constants. Suppose

$$x^4+x^2+1=(x-1)^4+A(x-1)^3+B(x-1)^2+C(x-1)+D$$

as polynomials. Find the values of A, B, C, D.

58. Let p, q, a be constants, with $a^2 \neq 1$, and $f(x)$ be the polynomial $x^4 + px^2 + qx + a^2$. Suppose $f(x)$ is divisible by $x^2 - 1$. Prove that $f(x)$ is divisible by $x^2 - a^2$.

59. Let $f(x)$ be a polynomial, and a, b, c be real numbers, with $a \neq 0$. Suppose that the remainder obtained upon dividing $f(x)$ by $x^2 - a^2$ is $bx + c$.

(a) Express b, c respectively in terms of a, $f(a)$, $f(-a)$.

(b) Further suppose that $f(-1) = 3$ and $f(1) = 1$. Find the remainder obtained upon dividing $f(x) = x^2 - 1$.

60. Let a, b, c be numbers. Suppose a, b, c are in an arithmetic progression. Prove that $2^a, 2^b, 2^c$ are in a geometric progression.

61. Let $x_1, x_2, x_3, \cdots$ be a geometric progression, (with the n-th term of being x_n). Suppose $x_1 = 40$ and $x_3 : x_7 = 2 : 3$. Find the value of x_{13}.

62. Let a, b, c, x, y, z be real numbers. Suppose a, b, c are in arithmetic progression and x, y, z are in geometric progression.

Prove that

$$(b - c)\ln(x) + (c - a)\ln(y) + (a - b)\ln(z) = 0.$$

63. Let $x > 0$. Suppose $x \neq 1$. Prove that:

(a) $x + \frac{1}{x} > 2$. (b) $x^2 + \frac{1}{x^2} > x + \frac{1}{x}$.

64. Suppose x, y are positive real numbers.

Prove that $\frac{1}{x^3} + \frac{1}{y^3} \geq \frac{x + y}{x^2y^2}$.

65. Let s, t, u, v be positive real numbers. Suppose $s : t = u : v$.

(a) Suppose s is the greatest amongst s, t, u, v. Prove that v is the least amongst s, t, u, v.

(b) Suppose s, v are respectively the greatest and the least amongst s, t, u, v. Prove that $s + v > t + u$.

Section 2: Trigonometry, plane coordinate geometry, and vector geometry.

- In many questions below, various geometric configurations are described in words. For such a question, you are advised to draw appropriate diagrams before proceeding with the calculation.

1. Let Γ be a circle with centre O and radius 6 units, and C be a point on Γ. Suppose BC be a line segment of length $2\sqrt{3}$ units which is tangent to Γ at C. Further suppose OB intersects the Γ at A.

   What is the area of the sector AOC?

2. Suppose $\triangle ABC$ is an equilateral triangle of side r. Three circles, each of radius r, are drawn with centre A, B, C respectively.

   Find the area of the region which is common to all of these three circles.

3. Let $\triangle AED$ be a triangle. Suppose $\angle AED$ is a right angle. Let ℓ be the line which passes through E and which is perpendicular to AD. Denote by C the point of intersection between ℓ and AD. Let ℓ' be the line which passes through C and which is perpendicular to AE. Denote by B the point of intersection between ℓ' and AE.

   Suppose $DE = a$ and $\angle DAE = \theta$. Express the length of BC in terms of a and θ.

4. Let ABC an isosceles triangle, with $AC = AB$. Let $PQRS$ be a square. Suppose P lies on AB, Q lies on AC, and both R, S lie on BC.

   Suppose $PQ = h$ and $\angle ABC = \theta$.

   Express AB in terms of h and θ.

5. Consider the trapezium $ABCD$, in which AB, CD are parallel line segments, and BC is perpendicular to each of AB, CD. Suppose $BC = s$, $CD = t$, and $\angle ADB = \theta$.

   Prove that the length of AB is given by

   $$\frac{(s^2+t^2)\sin(\theta)}{s\cos(\theta)+t\sin(\theta)}.$$

6. Verify that

   $$\frac{1}{\sin(\theta)+1} - \frac{1}{\sin(\theta)-1} = 2\sec^2(\theta)$$

   whenever $\sin^2(\theta) \neq 1$.

7. Let θ be a real number.

   Suppose $\cos(\theta) - \sin(\theta) = \sqrt{2}\sin(\theta)$.

   Prove that $\cos(\theta) + \sin(\theta) = \sqrt{2}\cos(\theta)$.

8. Let $\theta, \varphi, \zeta \in \mathbb{R}$.

   Suppose $\left(\frac{\tan(\theta)}{\sin(\zeta)} - \frac{\tan(\varphi)}{\tan(\zeta)}\right)^2 = \tan^2(\theta) - \tan^2(\varphi)$.

   Verify that $\cos(\zeta) = \frac{\tan(\varphi)}{\tan(\theta)}$.

9. Verify that

   $$\frac{\sin(3\theta)}{\sin(\theta)} + \frac{\cos(3\theta)}{\cos(\theta)} = 4\cos(2\theta)$$

   whenever θ is not an integral multiple of $\frac{\pi}{2}$.

10. Verify that $\sin(\theta) + \sin(\theta + \frac{2\pi}{3}) + \sin(\theta + \frac{4\pi}{3}) = 0$.

11. (a) Verify that

    $$\sin(\theta)+\sin(2\theta)+\sin(3\theta) = (M\cos(\theta)+N)\sin(M\theta).$$

    Here M, N are positive constants whose values you have to determine explicitly.

    (b) Hence, or otherwise, prove that

    $$\frac{\sin(\theta)+\sin(2\theta)+\sin(3\theta)}{\cos(\theta)+\cos(2\theta)+\cos(3\theta)} = \tan(2\theta),$$

    whenever $\cos(2\theta) \neq 0$.

12. Suppose $\tan(\theta) + \cot(\theta) = \frac{25}{12}$ and $\pi < \theta < \frac{3\pi}{2}$. Find the value of $\sin^3(\theta) + \cos^3(\theta)$.

13. Let $\alpha, \beta, \gamma, \delta$ be acute angles. Suppose $\tan(\alpha) = \frac{1}{3}$, $\tan(\beta) = \frac{1}{5}$, $\tan(\gamma) = \frac{1}{7}$ and $\tan(\delta) = \frac{1}{8}$.

    (a) Find the respective values of $\tan(\alpha + \beta)$ and $\tan(\gamma + \delta)$.

    (b) Find the value of $\alpha + \beta + \gamma + \delta$. (Leave your answer in radians.)

14. Suppose $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Further suppose $\alpha + \beta + \gamma = \frac{\pi}{2}$. Verify that

    $$\tan(\alpha)\tan(\beta) + \tan(\beta)\tan(\gamma) + \tan(\gamma)\tan(\alpha) = 1.$$

15. Solve the equation $\cos(2\theta) - \sqrt{3}\cos(\theta) + 1 = 0$. (Leave your answers in radians.)

16. Solve the equation $\sin(5\theta) + \sin(3\theta) = \cos(\theta)$. (Leave your answers in radians.)

17. (a) Express $\sin^2(\mu) - \sin^2(\nu)$ in terms of $\cos(\mu + \nu)$, $\cos(\mu - \nu)$, $\sin(\mu + \nu)$, $\sin(\mu - \nu)$.

    (b) Hence, or otherwise, solve the equation

    $$\sin^2(3\theta) - \sin^2(2\theta) - \sin(\theta) = 0.$$

18. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

    $$f(x) = \frac{2(\sin^4(x) - \cos^4(x) - 2)}{4\cos^2(x) + 5} \quad \text{for any } x \in \mathbb{R}.$$

    (a) Prove that $f(x) = \frac{3}{4\cos^2(x)+5} - 1$ for any $x \in \mathbb{R}$.

    (b) Hence, or otherwise, find the maximum and minimum values of the function f.

19. Suppose the lengths of BC, CA, AB in $\triangle ABC$ are given by a, b, c respectively.

    Further suppose $(b + c) : (c + a) : (a + b) = 5 : 6 : 4$.

    Find the ratio $\cos(A) : \cos(B) : \cos(C)$.

20. Suppose the lengths of BC, CA, AB in $\triangle ABC$ are given by a, b, c respectively.

    (a) Prove that

    $$a^2+b^2+c^2 = 2(bc\cos(A)+ca\cos(B)+ab\cos(C)).$$

    (b) Prove that $\dfrac{\tan(B)}{\tan(C)} = \dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}$.

21. The area of the triangle bounded by the two lines $x+y=4$ and $x-y=2p$ and the y-axis is 9.

    Find all positive values of p.

22. Find the ratio in which the line segment joining the points $A=(3,-1)$, $B=(-1,1)$ is divided by the straight line $x-y-1=0$.

23. Let a, p be non-zero constants, and $A=(a,0)$, $B=(ap^2, 2ap)$, $C=(\frac{a}{p^2}, -\frac{2a}{p})$.

    Prove that A, B, C are collinear.

24. Let ℓ be the line which cuts the x-axis at $A=(4,0)$ and the y-axis at $B=(0,6)$.

    (a) Find an equation for the line ℓ.

    (b) Let C be a point on the line segment AB, and M be the midpoint of the OC. Denote the coordinates of M by (p,q).

    Prove that $\frac{p}{2}+\frac{q}{3}=1$.

25. Let $A=(4,3)$, $B=(0,-5)$, $C=(-3,4)$. Consider the triangle $\triangle ABC$.

    (a) Find the equations of the perpendicular bisectors of the sides of $\triangle ABC$.

    (b) Prove that the three perpendicular bisectors are concurrent.

    (c) Find the equation of the circumscribed circle of $\triangle ABC$.

26. Let a, b, c, d be non-zero real numbers, with $a-b \neq 0$ and $a+b \neq 0$. Let ℓ_1 be the line given by the equation $ax+by+c=0$, and ℓ_2 be the line given by the equation $(a-b)x+(a+b)y+d=0$.

    Suppose θ is the acute angle between the lines ℓ_1, ℓ_2.

    What is the value of $\tan(\theta)$?

27. Let $A=(7,3)$, $B=(-1,5)$, and C be the mid-point of the line segment AB.

    (a) Find the coordinates of C.

    (b) Let k be a real number, and $P=(k^2,k)$ be a point outside the line segment AB.
    Suppose CP is perpendicular to AB.
    Find all possible values of k.

28. Let k be a real constant. Denote by L_k the straight line given by the equation $(2-k)x+(1+2k)y-(4+3k)=0$.

    Suppose the distance of L_k from the origin is equal to 1.

    Find all possible values of k.

29. Let L be the line given by the equation $x-7y+3=0$.

    Let a be a positive real number, and C be the circle given by the equation $(x-2)^2+(y+5)^2=a$.

    (a) Find the distance from the centre of C to L.

    (b) Suppose L is a tangent to C.
    Find the value of a.

30. Let A, B be the points $(1,2)$, $(2,0)$ respectively. Let r be a positive real number, and P be the point on the line segment AB satisfying the condition that $AP:PB=1:r$.

    (a) Express the coordinates of P in terms of r.

    (b) Show that the slope of the line OP is $\dfrac{2r}{2+r}$.

    (c) Suppose $\angle AOP = 45°$. Find the value of r.

31. Suppose PQ is the chord of the circle C with equation $x^2+y^2-6x-8y+21=0$. Further suppose the midpoint M of PQ is $(2,3)$.

    Find the coordinates of the points P, Q respectively.

32. Let C be the circle given by the equation $x^2+y^2+8x+12y-48=0$.

    (a) Find the centre and the radius of C.

    (b) Prove that the origin O is a point inside the circle.

    (c) Suppose O is the mid-point of some chord ℓ of the circle C. Find the equation of ℓ.

    (d) What is the length of the chord ℓ?

33. Let the circles C_1, C_2 be given by the equations $x^2+y^2+2x-4y+3=0$, $x^2+y^2-6x+4y-5=0$ respectively.

    (a) Find the radii and the centres of C_1, C_2 respectively.

    (b) Find the distance between the respective centres of C_1, C_2.

    (c) Hence, or otherwise, prove that C_1, C_2 touches each other.

34. Let $P=(6,3)$. Find the equations of all circles so that each touches both axes and passes through the point P.

35. Find all possible values of k for which the line $y=2x+k$ is tangent to the circle $x^2+y^2=5$.

36. Find all possible values of k for which the line $y=kx$ is tangent to the circle $x^2+y^2-4x+2y=0$.

37. The circle C given by the equation $x^2+y^2-10x+8y+16=0$ cuts the x-axis at $P=(p,0)$ and $Q=(q,0)$ and touches the y-axis at $T=(0,t)$. It is given that $p \leq q$.

    (a) What is the radius and the centre of C?

    (b) Find the respective values of p, q, t.

    (c) Let R be a point on the circle C. Suppose PR is parallel to TQ.

        i. Find the equation of PR.

ii. Find the coordinates of R by solving simultaneously the equation of PR and the eqaution of C.

38. Two circles C_1, C_2, with centres P_1, P_2 respectively, touch externally at the point Q (so that P_1 lies outside of the circle C_2, and P_2 lies outside the circle C_1). The equation of C_1 is given by $x^2 + y^2 - 6x - 6y + 16 = 0$. The point P_2 is $(-1, -1)$.

    (a) Find the equation of the circle C_2.

    (b) Find the equation of the common tangent L at Q.

    (c) Suppose L cuts the y-axis at the point R.
    Find the equation of the other tangent T from R to the circle C_2.

39. Let $A = (4, 0)$. Let P be a variable point (with coordinates (x, y)). Suppose P is equidistant from A and from the y-axis.

    Find an equation relating x, y which describes the locus of the point P.

40. Let P be a variable point (with coordinates (x, y)). Suppose P varies accoding to the equations

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}.$$

    (a) Prove that the locus of the point P is a circle.

    (b) Which point(s) on the circle does the variable point P never reach?

41. Find an equation (relating x, y) describing the locus of the variable point which varies according to each respective pair of parametric equations in t below.

    (a) $x = t + t^{-1}$, $y = t - t^{-1}$.

    (b) $x = t + 1$, $y = t^2 + t$.

    (c) $x = 3\cos(t)$, $y = 4\sin(t)$.

    (d) $x = 5\sec(t)$, $y = 12\tan(t)$.

    (e) $x = \sin(t) - \cos(t)$, $y = \csc(t) + \sec(t)$.

    (f) $x = \sin(t) + \cos(t)$, $y = \tan(t) + \cot(t)$.

42. Let p be a real constant, and $\mathbf{a} = p\mathbf{i}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{c} = 3\mathbf{i} - \mathbf{j}$.

    Find all possible values of p for which the points represented by the position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear.

43. Suppose $\mathbf{p} = \mathbf{i} + \mathbf{j}$, $\mathbf{q} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{r} = \mathbf{i} + 3\mathbf{j}$. Find some scalars μ, ν which satisfy $\mathbf{p} = \mu\mathbf{q} + \nu\mathbf{r}$.

44. Prove that the points with position vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}-2\mathbf{j}$, $3\mathbf{i} + 4\mathbf{j}$, $3\mathbf{i} + 7\mathbf{j}$ form a parallelogram.

45. Let Γ be a circle with centre O, and A, P, B be points on the circumference of Γ. Let Q be a point outside Γ.

    Suppose AOB is a diameter of Γ, and $\overrightarrow{OA} = -4\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{OP} = -5\mathbf{i}$ and $\overrightarrow{BQ} = 2\overrightarrow{BP}$.

    (a) Express $\overrightarrow{OB}$, $\overrightarrow{BQ}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{i}, \mathbf{j}$.

    (b) Show that $\overrightarrow{OP}$ is parallel to $\overrightarrow{AQ}$.

46. Let P, Q, R be points on the 'infinite' plane. Suppose $\angle PQR$ is a right angle. Suppose $PQ = 7$.

    What is the value of $\overrightarrow{PQ} \cdot \overrightarrow{PR}$?

47. The position vectors of A, B, C, D are $2\mathbf{i} + 4\mathbf{j}$ $5\mathbf{i} + 2\mathbf{j}$, $-2\mathbf{i} + 7\mathbf{j}$, $2\mathbf{i} + 13\mathbf{j}$ respectively.

    Prove that AB is perpendicular to CD.

48. Let $\mathbf{u}$, $\mathbf{v}$ be vectors. Suppose $|\mathbf{u}| = 4$ and $|\mathbf{v}| = 6$. Further suppose the angle between $\mathbf{u}$ and $\mathbf{v}$ is $60°$.

    Find the respective values of:

    (a) $|\mathbf{u} + \mathbf{v}|$ (b) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

49. Let c be a real number, and $\mathbf{u} = 4\mathbf{i} + (3 - c)\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + (c - 1)\mathbf{j}$. Suppose $\mathbf{u}, \mathbf{v}$ are perpendicular to each other.
    Find all possible values of c.

50. Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$. Suppose $\mathbf{b}$ is a vector satisfying $|\mathbf{b}| = 4$. Further suppose that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$.

    (a) Find $|\mathbf{a}|$.

    (b) Find $\mathbf{a} \cdot \mathbf{b}$.

    (c) Let m be a real constant. Suppose that the vector $m\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{b}$.
    Find the value of m.

51. Let A, B, C be points on the 'infinite' plane. Suppose the position vectors of A, B, C are given by $2\mathbf{i} + 7\mathbf{j}$, $8\mathbf{i} + \mathbf{j}$ and $-\mathbf{i} - 2\mathbf{j}$ respectively.

    (a) Suppose M is the mid-point of the line segment AB. Express $\overrightarrow{OM}$ in terms of $\mathbf{i}, \mathbf{j}$.

    (b) Suppose θ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
    Find the value of $\cos(\theta)$.

52. Let A, B, C, D be points on the 'infinite' plane. Suppose P, Q are the mid-points of the line segments AC, BD respectively.

    Prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{PQ}$.

53. Let Γ be a circle with centre O. Suppose AOB is a diameter. Let C be a point on the circumference of Γ. Suppose $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OC} = \mathbf{v}$.

    (a) Express $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\mathbf{u}, \mathbf{v}$

    (b) Find the value of $\overrightarrow{AC} \cdot \overrightarrow{BC}$.

    (c) Hence deduce that $\angle ACB$ is a right angle.

54. Consider a triangle $\triangle ABC$. Suppose AD, BE are medians of this triangle. Suppose AD and BE meet at the point G. Write $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{CA} = \mathbf{c}$.

    (a) Express $\overrightarrow{AB}$, $\overrightarrow{BE}$ respectively in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

    (b) Prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \mathbf{0}$.

55. Let $\mathbf{m}, \mathbf{n}$ be vectors in $\mathbb{R}^3$ and λ be a real number. Suppose $\mathbf{u} = \lambda\mathbf{m} + (1 - \lambda)\mathbf{n}$ and $\mathbf{v} = 2(1 - \lambda)\mathbf{m} - \lambda\mathbf{n}$.

    (a) Verify that $\mathbf{u} \times \mathbf{v} = (-3\lambda^2 + 4\lambda - 2)\mathbf{m} \times \mathbf{n}$.

    (b) Further suppose $|\mathbf{m}| = 3$, $|\mathbf{n}| = 4$, and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\frac{\pi}{6}$.

    Find the value of $|\mathbf{m} \times \mathbf{n}|$. Hence, or otherwise, determine the smallest possible area of the parallelogram with adjacent sides $\mathbf{u}$, $\mathbf{v}$.

Section 3: Differentiation and integration.

1. Find $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$ from first principles.

2. Find $\frac{d}{dx}(\tan(x))$ from first principles.

3. Find $\frac{d}{dx}(xe^x)$ from first principles.

4. Find $\frac{dx}{dy}$ in each of the relations below:

   (a) $y = 4x - x^3$

   (b) $y = \sqrt{\frac{x+1}{x+2}}$.

   (c) $y = \sin(2x)$.

   (d) $y = \frac{e^x - e^{-x}}{2}$.

   In parts (c), (d), express your answers in terms of y.

5. Let C be the curve given by the equation

$$y^2 + x^{\frac{1}{3}}y - 3 = 0.$$

   Find the value of $\frac{dy}{dx}$ at the point $P = (8, 1)$.

6. Let C be the curve given by the equation

$$x^2 + xy - y^2 = 1.$$

   Prove that $\frac{d^2y}{dx^2} = \frac{P}{(x + Qy)^R}$.

   Here P, Q, R are integers whose respective values you have to determine explicitly.

7. Let c be a positive real number, and $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \frac{x}{c}\cos(\frac{c}{x}) \text{ for any } x \in \mathbb{R}\backslash\{0\}.$$

   Prove that $x^4 f''(x) + Ac^2 f(x) = B$ for any $x \in \mathbb{R}\backslash\{0\}$.

   Here A, B are integers whose respective values you have to determine explicitly.

8. Let C be the curve given by $y = x + \frac{1}{x}$.

   Find the equations of the tangent and the normal to C at the point $(1, 2)$.

9. Let p, q, r be real numbers, and C be the curve given by the equation $y = px^3 + qx^2 + rx$.

   Suppose the tangents to C at $x = 1$ and at $x = -1$ respective are parallel to the x-axis. Further suppose the slope of the normal to C at $x = 0$ is $-\frac{1}{6}$.

   Find the values of p, q, r respectively.

10. Let E be the curve given by the equation

$$\frac{x^2}{2} + \frac{y^2}{7} = 1.$$

   Let $L_{m,c}$ be the line given by the equation $y = mx+c$.

   (a) Suppose the line $L_{m,c}$ is a tangent to the curve E. Show that $c^2 = 2m^2 + 7$.

   (b) Suppose $L_{m,c}$ passes through the point $(0, 5)$. Find the value(s) of m, c respectively.

11. Let a, b be non-zero real numbers. Let C_1, C_2 be the curves given by the equations $x^2 + 2y^2 = 6$ and $ax^2 - by^2 = 3$.

   Suppose C_1 and C_2 intersect at some point P, and their respective tangents at P are perpendicular to each other.

   Prove that $\frac{1}{a} + \frac{1}{b} = 1$.

12. The base of a right-angled triangle is 12cm. If the height of the triangle is increasing at the rate of 3cm per second, how fast is its hypotenuse chaging when the height is 5cm?

13. Salt is being dumped from a pipe at a rate of 3m^3 per minute. It forms a cone whose height is equal to the radius of the base. Find the rate at which the height of the cone increases when the height is 3m.

14. Let $P = (3, 0)$, and C be the curve given by the equation $y^2 = x^2 + 2$. Find the shortest distance from the point P to the curve C.

15. Let Γ be a circle with centre O and radius r. Suppose an isosceles triangle $\triangle ABC$ with $AB = AC$ is inscribed in Γ. Denote by θ the angle $\angle OAB$.

   (a) Express the length of BC in terms of r and θ.

   (b) Prove that the area of $\triangle ABC$ is given by $4r^2 \sin(\theta)\cos^3(\theta)$.

   (c) (Suppose B, C are variable points, but A is fixed.) Determine, in terms of the value of θ, when the area of $\triangle ABC$ attains maximum.

16. Find $\int \sqrt{2x+1}\, dx$.

17. Find $\int x\sqrt{x-1}\, dx$.

18. Find $\int x\ln(x)\, dx$.

19. Find $\int \sin^2(3x)dx$.

20. Find $\int \tan^2(4x)dx$.

21. Evaluate the definite integral $\int_0^{\frac{\pi}{3}} \tan^4(x)dx$.

22. Evaluate the definite integral $\int_0^{\pi/2} \cos^5(\theta)\sin^2(\theta)d\theta$.

23. Evaluate the definite integral $\int_e^{e^2} \frac{\ln(\ln(u))}{u\ln(u)}du$.

24. (a) Prove that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3(x)dx}{\cos(x)+\sin(x)} = \int_0^{\frac{\pi}{2}} \frac{\cos^3(x)dx}{\cos(x)+\sin(x)}.$$

(b) Hence evaluate both definite integrals.

25. (a) Prove that

$$\int_0^a f(t)dt = \frac{1}{2}\int_0^a (f(t)+f(a-t))dt.$$

(b) Hence, or otherwise, prove that

$$\int_0^{\pi/4} \ln(1+\tan(t))dt = \frac{\pi\ln(2)}{8}.$$

26. The slope at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = \tan^3(x)\sec(x).$$

Suppose the curve passes through the origin.

Find its equation.

27. Let C be the curve given by the equation

$$y = \frac{1}{1+\cos(x)} \text{ for } 0 \leq x < \pi.$$

Find the area of the region bounded the curve C, the x-axis and the lines $x = 0$, $x = \frac{\pi}{2}$.

28. Let C be the curve given by the equation

$$y^2 = x(1-x)^2.$$

(a) Find the points P, Q where C intersect the x-axis.

(b) Find the area of the region bounded by the curve C between P and Q.

29. Let C_1, C_2 be the curves given repectively by the equations $y = x^3$, $y = x^3 - 6x^2 + 12x$.

(a) Find all the points of intersection of C_1 and C_2.

(b) Find the area of the region bounded by C_1 and C_2.

30. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is rotated about the x-axis through 360°. What is the volume of the solid so formed?