1 (i)

Since $\lim_{t \to 0^+} \frac{h(t)}{t} = 0$. For any $\varepsilon > 0$, there exist $\delta > 0$, such that for $t \in (0, \delta)$, we have $|h(t)| < t\,\varepsilon$.

When $t = \frac{1}{q}$, we write $|h(\frac{1}{q})| < \frac{\varepsilon}{q}$ for $\frac{1}{q} < \delta$.

If $x = \frac{p}{q} \in N(\varepsilon)$, we have

$$|\varphi(x)| = |p\, h(\tfrac{1}{q})| \geq \varepsilon.$$

Note that $0 < \frac{p}{q} < 1$, then $|h(\frac{1}{q})| \geq \frac{\varepsilon}{p} > \frac{\varepsilon}{q}$

Therefore, for $|\varphi(x)| \geq \varepsilon$, we must have $\frac{1}{q} > \delta$, i.e. $q < \frac{1}{\delta}$. This means that there is an upper bound on the possible value of q. It follows that there are finite number of p, such that $\frac{p}{q}$ is relatively prime.

Hence $N(\varepsilon)$ is a finite set.

(ii) Let x_0 be an irrational point in $(0, 1)$. Then $\varphi(x_0) = 0$. To show φ is continous at x_0 is to show $\lim_{x \to x_0} \varphi(x) = 0$.

By (i) $N(\varepsilon)$ is finite. we denote points in $N(\varepsilon)$ by

$x_1, x_2, x_3, \cdots, x_k$.

We set $\delta = \min_{j=1,2,\cdots,k} |x_0 - x_j| > 0$.

For any $\varepsilon > 0$, if x is rational and $|x - x_0| < \delta$.

It is clear that $x \notin N(\varepsilon)$ and $|\varphi(x)| < \varepsilon$

If x is irration then $\varphi(x) = 0$.

Hence, for any x satisfies $|x - x_0| < \delta$, we must have

$$|\varphi(x)| < \varepsilon.$$

Since fore every $\varepsilon > 0$ and every irration point x_0, we can find $\delta > 0$ such that $|\varphi(x) - \varphi(x_0)| < \varepsilon$ when $|x - x_0| < \delta$. Hence the function φ is continuous at every irrational point in $(0,1)$.

2. (i)

For any $x \in A$, we have

$$q(x) = \sqrt[3]{|x_1|^3 + \cdots + |x_m|^3} = 1$$

i.e. $$|x_1|^3 + |x_2|^3 + \cdots + |x_m|^3 = 1$$

It follows that $|x_i| \le 1$, $i = 1, 2, \cdots, m$.

then $\|x\| = \sqrt[2]{x_1^2 + x_2^2 + \cdots + x_m^2} \le \sqrt{m}$.

Hence A is bounded.

It is clear that $q: \mathbb{R}^m \to \mathbb{R}$ is continuous and $\{1\}$ is a closed set. Then A is closed.

A is closed and bounded, so A is compact.

(ii) Let $y_i = \dfrac{|x_i|^2}{\|x\|^2}$. then $\sum y_i = 1$ and $0 \le y_i \le 1$ (since $|x_i|^2 \le \|x\|^2$)

$$q(x)^3 = \sum |x_i|^3 = \sum (|x_i|^2)^{\frac{3}{2}} = \|x\|^3 \sum y_i^{\frac{3}{2}}$$

Note that $y_i^{\frac{3}{2}} \le y_i$ (because $0 \le y_i \le 1$) and $\sum y_i^{\frac{3}{2}} \le \sum y_i = 1$

Hence $q(x)^3 \le \|x\|^3$ and $q(x) \le \|x\|$.

By Hölder Inequality $\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}$, $\frac{1}{p} + \frac{1}{q} = 1$

with $p = 3$ and $q = \frac{3}{2}$, we have

$$\sum_{i=1}^{m} |x_i|^2 = \sum_{i=1}^{m} 1 \cdot |x_i|^2 \le \left(\sum_{i=1}^{m} 1^3\right)^{\frac{1}{3}} \left(\sum (|x_i|^2)^{\frac{3}{2}}\right)^{\frac{2}{3}} = m^{\frac{1}{3}} \left(\sum |x_i|^3\right)^{\frac{2}{3}}$$

$$= m^{\frac{1}{3}} q(x)^2$$

It follows $\|x\| \le m^{\frac{1}{6}} q(x)$.

Therefore $c_1 = 1$, $c_2 = m^{\frac{1}{6}}$.

3. (i) Note that A is closed and bounded, so A is compact set.

Let $B = \{\frac{1}{n}, n=1,2,\cdots\}$. B is not compact, because the limit point $0 \notin B$.

The image of a compact set under a continuous function is compact. Hence there is no continuous function f defined on A with image B.

(ii)

For any fixed $n \geq 1$. choose $n-1$ distinct irrational number $0 < b_1 < b_2 < \cdots < b_{n-1} < 1$. Define the sets

$$S_k = (b_{k-1}, b_k) \cap \mathbb{Q}, \text{ for } k=1,\cdots,n$$

where $b_0 = 0$, $b_n = 1$. Eeach S_k is both closed and open. These S_k are disjoint, non-empty and their union is D.

Define $g: D \to \mathbb{R}$ by $g(x) = k$ for $x \in S_k$, $k=1,2,\cdots n$.

Then $g(D) = \{1, 2, \cdots, n\}$.

To verify continuity of g: For every $x \in S_k$, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subset (b_{k-1}, b_k)$

Thus $(x-\delta, x+\delta) \cap D \subset S_k$. So g is a constant on this neighborhood, hence g is continuous at x.

4. (i)

For any $\varepsilon > 0$, we set $\delta = \frac{\varepsilon}{c}$. If $|u-v| < \frac{\varepsilon}{c}$ then $|f(u) - f(v)| < \varepsilon$.

(ii)

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

It is clear that f is continuous on $(0,1]$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x \sin \tfrac{1}{x}| \le \lim_{x\to 0^+} |x| = 0.$$

then f is continous on $[0,1]$ and hence uniforly continuous.

Let $u_k = \frac{1}{2k\pi + \frac{\pi}{2}}$, $v_k = \frac{1}{2k\pi + \frac{3\pi}{2}}$

$$f(u_k) = \frac{1}{2k\pi + \frac{\pi}{2}} \qquad f(v_k) = -\frac{1}{2k\pi + \frac{3\pi}{2}}$$

Then $|f(u_k) - f(v_k)| = \frac{1}{2k\pi + \frac{\pi}{2}} + \frac{1}{2k\pi + \frac{3\pi}{2}}$

and $|u_k - v_k| = \dfrac{\pi}{(2k\pi + \frac{\pi}{2})(2k\pi + \frac{3\pi}{2})}$

$$\frac{|f(u_k) - f(v_k)|}{|u_k - v_k|} = \frac{\dfrac{4k\pi + 2\pi}{(2k\pi + \frac{\pi}{2})(2k\pi + \frac{3\pi}{2})}}{\dfrac{\pi}{(2k\pi + \frac{\pi}{2})(2k\pi + \frac{3\pi}{2})}} = 4k + 2$$

when $k \to \infty$, $|u_k - v_k| \to 0$ but $\dfrac{|f(u_k) - f(v_k)|}{|u_k - v_k|} \to \infty$

It follows that there is not such c such that

$$\lim_{k \to \infty} \frac{f(u_k) - f(v_k)}{|u_k - v_k|} \le c.$$