Exercise 2.13. *Let A be a non-empty subset of X. A point $a \in X$ is called a boundary point of A if $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$ for all $r > 0$, where A^c denotes the complement of A in X. The set of all boundary points, write ∂A, of A is called the boundary of A.*

(i) *Find the boundaries of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}$.*
(ii) *Let $X = (0,1) \cup (2,3)$. Find the boundary of the set $(0,1)$ in X.*
(iii) *Show that the boundary ∂A is a closed subset of X.*
(iv) *Show that $\overline{A} = A \cup \partial A$.*

(ii) Let $A = (0,1)$. Then $A^c = (2,3)$ in X.

For any $a \in (0,1)$ and $r < 2-a$, $B(a,r) \cap A \neq \phi$, but $B(a,r) \cap A^c = \phi$.

Similarly, for any $b \in (2,3)$ and $r < b-1$, we have $B(b,r) \cap A = \phi$.

Hence $\partial A = \phi$ in X.

(iii)

By the Definition 2.8. ∂A is closed in X if $\overline{\partial A} = \partial A$.

Using Definition 2.4, $\overline{\partial A} = \partial A \cup D(\partial A)$ where $D(\partial A)$ denote the set of all limit points in of ∂A. i.e.

$$D(\partial A) = \{ x \in X : B(x,r) \setminus \{x\} \cap \partial A \neq \phi \text{ for all } r>0 \}.$$

Thus it suffices to show that $D(\partial A) \subseteq \partial A$. Let $x \in D(\partial A)$.

Then $B(x,r) \cap \partial(A) \neq \phi$ for all $r>0$.

For any $r>0$, there exists $x_r \in B(x, \frac{r}{2}) \cap \partial A$, which satisfies $B(x_r, s) \cap A \neq \phi$, $B(x_r, s) \cap A^c \neq \phi$ for all $s>0$.

Since $B(x_r, \frac{r}{2}) \subseteq B(x,r)$, then $B(x,r) \cap A \neq \phi$, $B(x,r) \cap A^c \neq \phi$, for all $r>0$. Thus $x \in \partial A$ and ∂A is closed.

(iv) For $x \in \partial A \setminus A$, we have $(B(x,r) \setminus \{x\}) \cap A \neq \phi$, $\forall r>0$,

Then $x \in D(A)$ and hence $\partial A \setminus A \subseteq D(A)$. It follows that

$$A \cup \partial A \subseteq A \cup D(A) = \overline{A}$$

On the other hand, if $x \in D(A) \setminus A$, we have $(B(x,r) \setminus \{x\}) \cap A \neq \phi$ and $B(x,r) \cap A^c \neq \phi$ for all $r>0$.

which implies $D(A) \setminus A \subseteq \partial A$. Hence $\overline{A} = A \cup D(A) \subseteq A \cup \partial A$.

Exercise 2.17. *(i) Let V be a subset of X. A point $z \in V$ is said to be an interior point of V if there is $r > 0$ such that $B(z,r) \subseteq V$. If we put int(V) the set of all interior points of V, show that int(V) is an open subset of X.*

(ii) A metric d on X is said to be non-archimedean if it satisfies the strong triangle inequality, that is, $d(x,y) \leq \max(d(x,z), d(z,y))$ for all x, y and $z \in X$ (see also Example 1.2 (iv)). Show that if d is a non-archimedean metric on X, then for every closed ball $\overline{B}(a,r) := \{x \in X : d(a,x) \leq r\}$ is an open set in X.