IMSC 2058 Solution for Homework 9

Ex 6.12

Assume X is normal. Since F and $X \setminus V$ are disjoint closed sets, there are disjoint open sets W and O for which $F \subseteq W$ and $X \setminus V \subseteq O$. Thus $W \subseteq X \setminus O \subseteq V$. Since $W \subseteq X \setminus O$ and $X \setminus O$ is closed, then $\overline{W} \subseteq X \setminus O \subseteq V$. Therefore $F \subseteq W \subseteq \overline{W} \subseteq V$.

To prove the converse, we let A, B be two disjoint closed subset of X. Then $X \setminus B$ is open and $A \subseteq X \setminus B$. Thus there is an open set $\mathcal{O}$ for which $A \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq X \setminus B$. Hence $\mathcal{O}$ and $X \setminus \overline{\mathcal{O}}$ are disjoint open neighborhoods of A and B.

Ex 7.6

No, the space $(C_c(X), \|\cdot\|_\infty)$ is not complete for a general locally compact space X.

Let X be a locally compact space but not compact space. Let $C_0(X)$ denote the set of all continuous functions $X \to \mathbb{R}$ (or $\mathbb{C}$) that vanish at infinity, i.e., for every $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. We can show that $(C_0(X), \|\cdot\|_\infty)$ is complete and $C_c(X)$ is a proper dense subset of $C_0(X)$. Thus, there exist Cauchy sequences in $C_c(X)$ converging (in the sup-norm) to limits in $C_0(X) \setminus C_c(X)$, so the limit lies outside $C_c(X)$.

We skip the proof of $(C_0(X), \|\cdot\|_\infty)$ is complete here. Next we show $C_c(X)$ is dense in $C_0(X)$.

Let $f \in C_0(X)$, and $K = \{x \in X : |f(x)| > \varepsilon/2\}$, which is compact. Thus, $|f(x)| < \varepsilon/2$ for all $x \notin K$. Let $L = \{x \in X : |f(x)| \geq \varepsilon\}$, then $L \subseteq K$. By Proposition 7.7., X admits a continuous function $\phi : X \to [0,1]$ with $0 \leq \phi \leq 1$ on X such that $\phi(x) = 1$ for all $x \in L$, and $\phi(x) = 0$ for all $x \in X \setminus K$.

Let $g = \phi f$. Then $g \in C(X)$ and $\text{supp}(g) \subseteq \text{supp}(\phi) \subseteq K$, which is compact, so $g \in C_c(X)$.

For $x \in L$, $\phi(x) \equiv 1$, so $g = f$ and $|f - g| = 0 < \varepsilon$.

For $\in K \setminus L$, $0 \leq \phi \leq 1$, so $|f(x) - g(x)| = |f(x)| \cdot |1 - \phi(x)| \leq |f(x)| < \varepsilon$.

For $x \in X \setminus K$, $\phi \equiv 0$, so $g \equiv 0$ and $|f - g| = |f| < \varepsilon/2 < \varepsilon$.

Thus, $|f(x) - g(x)| < \varepsilon$ for all $x \in X$, so $\|f - g\|_\infty < \varepsilon$. Since $\varepsilon > 0$ is arbitrary here, $C_c(X)$ is dense in $C_0(X)$.

Since X is non-compact, it cannot be covered by finitely many compact sets. In particular, there exists a discrete sequence $(p_n)_{n \in \mathbb{N}}$ of distinct points in X with no limit point in X. For each n, there exist disjoint open neighborhoods U_n of p_n. By Proposition 7.3, there exist compact $\overline{V_n}$ such that $\overline{V_n} \subseteq U_n$ for all n. Applying Proposition 7.7 again, we define $\phi_n : X \to [0,1]$ such that $\phi_n(p_n) = 1$, $\phi_n(X \setminus U_n) = 0$. Then $\text{supp}(\phi_n) \subseteq \overline{V_n}$. Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \phi_n(x).$$

Then $f \in C_0(X)$ but $f \notin C_c(X)$, because the support $\text{supp}(f) = \bigcup_{n=1}^{\infty} \text{supp}(\phi_n)$ is an infinite disjoint union of compacts with no limit point (by construction of (p_n)), which is not compact.