IMSC 2058 Solution for Homework 10

Ex 7.9

We first show that $\mathcal{T}_\infty$ is a topology.

Since $\emptyset \in \mathcal{T}$, then $\emptyset \in \mathcal{T}_\infty$. Note that $\emptyset$ is compact,then $X_\infty = X_\infty \setminus \emptyset \in \mathcal{T}_\infty$.

Let $\{V_\alpha\}_{\alpha\in A} \subseteq \mathcal{T}_\infty$. If every V_α does not contain ∞, $V_\alpha \cap X \in \mathcal{T}$, then $\bigcup V_\alpha \in \mathcal{T}$, so $\bigcup V_\alpha \in \mathcal{T}_\infty$. If V_{α_0} contains ∞, i.e. $V_{\alpha_0} = X_\infty \setminus C$ for compact $C \subseteq X$. Then

$$\bigcup_{\alpha\in A} V_\alpha = V_{\alpha_0} \cup \left(\bigcup_{\alpha\neq\alpha_0} V_\alpha\right) = (X_\infty \setminus C) \cup \left(\bigcup_{\alpha\neq\alpha_0} V_\alpha\right).$$

Since

$$X_\infty \setminus \bigcup_{\alpha\in A} V_\alpha = X_\infty \setminus \left[(X_\infty \setminus C) \cup \left(\bigcup_{\alpha\neq\alpha_0} V_\alpha\right)\right] = C \setminus \bigcup_{\alpha\neq\alpha_0} V_\alpha = C \setminus \bigcup_{\alpha\neq\alpha_0} (V_\alpha \cap C),$$

and $V_\alpha \cap C$ is open in the subspace topology on C. Then $C \setminus \bigcup_{\alpha\neq\alpha_0}(V_\alpha \cap C)$, is closed in C. Let $D = C \setminus \bigcup_{\alpha\neq\alpha_0}(V_\alpha \cap C)$. Since C is compact, D is compact. It follows that $\bigcup_{\alpha\in A} V_\alpha = X_\infty \setminus D$, hence $\bigcup_{\alpha\in A} V_\alpha \in \mathcal{T}_\infty$.

Let $V_1, \ldots, V_n \in \mathcal{T}_\infty$.

Case 1: If none of V_i contain ∞, then $(\bigcap V_i) \cap X = \bigcap (V_i \cap X) \in \mathcal{T}$, so $\bigcap V_i \in \mathcal{T}_\infty$.

Case 2: If $V_1 = X_\infty \setminus C$, where C is compact. Then

$$\bigcap V_i = (X_\infty \setminus C) \cap \bigcap_{i=2}^{n} V_i = X_\infty \setminus (C \setminus \bigcap_{i=2}^{n} (V_i \cap C)).$$

Again, $\bigcap_{i=2}^{n}(V_i \cap C)$ is open in C, so its complement in C is closed hence compact. Therefore $\bigcap V_i \in \mathcal{T}_\infty$.

Case 3: If $V_1 = X_\infty \setminus C_1$, $V_2 = X_\infty \setminus C_2$ where C_1, C_2 are compact sets. Then $V_1 \cap V_2 = X_\infty \setminus (C_1 \cup C_2)$, and $C_1 \cup C_2$ compact. Applying above discussion in Case 2 to derive $\bigcap V_i \in \mathcal{T}_\infty$. Similar process applied to finite sets contain ∞.

Thus, $\mathcal{T}_\infty$ is a topology.

Next, we show $(X_\infty, \mathcal{T}_\infty)$ is Compact and Hausdorff.

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X_∞. There must exists U_{α_0} such that $U_{\alpha_0} = X_\infty \setminus C$ for some compact $C \subseteq X$. Note that $X_\infty = (X_\infty \setminus C) \cup C$. To find the finite cover of X_∞, we only need to find finite cover of C. Now, $\{U_\alpha \cap C\}_{\alpha\in A}$ covers the compact $C \subseteq X$. By compactness of C, there exist $U_{\alpha_1}, \ldots, U_{\alpha_k}$, cover C. Then $\{U_\alpha\}_{\alpha\in A}$ has finite subcover $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ covers X_∞.

It is clear that any two distinct points $x, y \in X$ have disjoint open neighborhoods because X is Hausdorff. We consider the separation of $x \in X$, $\infty \notin X$. By Proposition 7.3, for any point $x \in X$ and any open neighborhood V of x, there exists a open neighborhood K such that

$x \in K \subseteq \overline{K} \subseteq V$. Then Then $W = X_\infty \setminus \overline{K} \in \mathcal{T}_\infty$, and $W \cap K = \emptyset$. Thus, $x \in X$ and $\infty \notin X$ have disjoint open neighborhoods in $\mathcal{T}_\infty$.

At last, we show X is Dense in $(X_\infty, \mathcal{T}_\infty)$.

To show density, it suffices to show every nonempty open set in $\mathcal{T}_\infty$ intersects X. Let U be a nonempty open set in $\mathcal{T}_\infty$. If $U \in \mathcal{T}$, then $U \cap X = U \in \mathcal{T}$. If U contains ∞, i.e. $U = X_\infty \setminus C$ for compact $C \subseteq X$. Then

$$U \cap X = (X_\infty \setminus C) \cap X = X \setminus C.$$

We claim that $X \setminus C \neq \emptyset$. If $X \setminus C = \emptyset$, we must have $C = X$. Then $U = X_\infty \setminus X = \{\infty\}$. But $\{\infty\}$ is open in $\mathcal{T}_\infty$ iff X is compact. It gives contradiction (X is locally compact nut not compact). Hence $U \cap X \neq \emptyset$ and X is Dense in $(X_\infty, \mathcal{T}_\infty)$.