IMSC2058 ANALYSIS I (2025-26, 1^{st} TERM)

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1. Metric Spaces

Throughout the note, we use the following notation:

(i) $\mathbb{R}$ = the set of all real numbers.
(ii) $\mathbb{C}$ = the set of all complex numbers.
(iii) $\mathbb{Q}$ = the set of all rational numbers.
(iv) $\mathbb{N}$ = the set of all natural numbers.
(v) $\mathbb{Z}$ = the set of all integers.

Definition 1.1. *Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if it satisfies the following conditions.*

(i) $d(x, y) \geq 0$ for all $x, y \in X$.
(ii) $d(x, y) = 0$ if and only if $x = y$.
(iii) (Symmetric property) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iv) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this case, the pair (X, d) is called a metric space.

Example 1.2. *:*

(i) For $x, y \in \mathbb{R}$, put $d(x, y) = |x - y|$. Then d is a metric on $\mathbb{R}$ and d is called the usual metric on $\mathbb{R}$.
(ii) For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, define:
$d_\infty(x, y) = \max(|x_1 - x_2|, |y_1 - y_2|)$;
$d_1(x, y) = |x_1 - x_2| + |y_1 - y_2|$;
$d_2(x, y) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$. *Then all are metrics on $\mathbb{R}^2$.*
(iii) Let X be any non-empty set. For $x, y \in X$, let $d(x, y) = 0$ if $x = y$; otherwise, $d(x, y) = 1$. Then d is a metric on X. In this case, d is called the discrete metric on X and (X, d) is called a discrete metric space.
(iv) Fix a prime number p. For $\frac{a}{b} \in \mathbb{Q}$, define $|\frac{a}{b}|_p = p^{-v}$ if $\frac{a}{b} = p^v \frac{a'}{b'}$ where $v \in \mathbb{Z}$ and $p \nmid a'b'$. If we put $d_p(x, y) = |x - y|_p$ for $x, y \in \mathbb{Q}$, then d_p is a metric on $\mathbb{Q}$. Furthermore, d_p satisfies the strong triangle inequality, i.e.,

$$d_p(x, y) \leq \max(d_p(x, z), d_p(z, y))$$

for all $x, y, z \in \mathbb{Q}$.

Definition 1.3. *Let V be a vector space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : V \to \mathbb{R}$ is called a norm on V if it satisfies the following conditions.*

(i) $\|x\| \geq 0$ for all $x \in V$.
(ii) $\|x\| = 0$ if and only if $x = 0$.
(iii) (Triangle inequality) $\|x - y\| \leq \|x - z\| + \|z - y\|$ for all $x, y, z \in V$.

In this case, the pair $(V, \|\cdot\|)$ is called a normed space.

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Proposition 1.4. *Let $(V, \|\cdot\|)$ be a normed space. If we put $d(x,y) = \|x-y\|$ for $x, y \in V$, then d is a metric on V. Consequently, every normed space is a metric space.*

Remark 1.5. *Let V be a vector space. Notice that the discrete metric d on V must not be induced by a norm, i.e., we cannot find a norm $\|\cdot\|$ on V such that $d(x,y) = \|x-y\|$ for $x, y \in V$.*

Example 1.6. *The following are important examples of normed spaces.*

(i) Let $\ell^\infty = \{(x_n) : x_n \in \mathbb{C},\ n = 1,2...;\ \sup |x_n| < \infty\}$ and $c_0 = \{(x_n) \in \ell^\infty :\ \lim |x_n| = 0\}$. Put $\|(x_n)\|_\infty = \sup |x_n|$.

(ii) Let $\ell^1 = \{(x_n) : x_n \in \mathbb{C},\ n = 1,2...;\ \sum_{n=1}^\infty |x_n| < \infty\}$. Put $\|(x_n)\|_1 = \sum_{n=1}^\infty |x_n|$.

(iii) Let $\ell^2 = \{(x_n) : x_n \in \mathbb{C},\ n = 1,2...;\ \sum_{n=1}^\infty |x_n|^2 < \infty\}$. Put $\|(x_n)\|_2 = \sqrt{\sum_{n=1}^\infty |x_n|^2}$.

(iv) Let $C[a,b] := \{f : [a,b] \to \mathbb{R} : f$ *is continuous*$\}$. For each $f \in C[a,b]$, put

$$\|f\|_\infty \sup_{x\in[a,b]} |f(x)|.$$

Then $(C[a,b], \|\cdot\|)$ is a normed space.

(v) Let $R[a,b] := \{f : [a,b] \to \mathbb{R} : f$ *is Riemann integrable*$\}$. Notice that $C[a,b] \subseteq R[a,b]$. For each $f \in R[a,b]$, put

$$\|f\|_1 := \int_a^b |f(x|dx.$$

Then $\|\cdot\|_1$ is NOT a norm function on $R[a,b]$. However, it is a norm function on $C[a,b]$.

Exercise 1.7. *:*

(1) Let (X,d) be a metric space. Define

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

for $x, y \in X$. Show that ρ is also a metric on X.

(2) Let (X, d_X) and (Y, d_Y) be the metric spaces. Define

$$\rho((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$$

for $x, x' \in X$ and y, y' in Y. Show that ρ is a metric on the product space $X \times Y = \{(x,y) : x \in X; y \in Y\}$.

(3) Let (X,d) be a metric space and let A be a subset of X. We say that A is bounded if there is $M > 0$ such that $d(a,a') \le M$ for all a, a' in A.
Show that if $A_1, ..., A_N (N < \infty)$ all are bounded subsets of X, show that $A_1 \cup \cdots\cdots \cup A_N$ is also a bounded subset of X.

2. Convergent Sequences

Throughout this section, (X, d) will denote a metric space.
For $a \in X$ and $r > 0$, put
$B(a, r) = \{x \in X : d(a, x) < r\}$, called the *open ball* with center a of radius r;
$\overline{B}(a, r) = \{x \in X : d(a, x) \le r\}$, called the *closed ball* with center a of radius r.

Recall that a sequence on X is a function $f : \{1, 2, ..\} \to X$. Write $f(n) = x_n \in X$. Also, if (n_k) is a sequence of positive integers with $n_1 < n_2 < n_3 < \cdots\cdots$, then we call (x_{n_k}) a subsequence of (x_n).

Definition 2.1. *A sequence (x_n) is said to be convergent in X if there is an element $a \in X$ such that $d(a, x_n) \to 0$ as $n \to \infty$, i.e., it satisfies the following condition.*

For any $\varepsilon > 0$, there is a positive integer N such that $x_n \in B(a, \varepsilon)$ for all $n \ge N$.

In this case, a is called a limit of the sequence (x_n). Also (x_n) is said to be divergent if it is not convergent

Proposition 2.2. *If (x_n) is a convergent sequence in X, then its limit is unique. Now we can write $\lim x_n$ for the limit of (x_n).*

Proof. Suppose that a and b both are the limits of (x_n) with $a \ne b$ in X. Then $d(a, b) > 0$. Choose $0 < 2\varepsilon < d(a, b)$. By the definition of limit, we can find the integers N_1 and N_2 such that $d(a, x_n) < \varepsilon$ for all $n \ge N_1$ and $d(b, x_n) < \varepsilon$ for all $n \ge N_2$. Now if we take $N \ge \max(N_1, N_2)$, then we have

$$d(a, x_N) < \varepsilon; \text{ and } d(b, x_N) < \varepsilon.$$

Hence we have

$$d(a, b) \le d(a, x_N) + d(x_N, b) < 2\varepsilon < d(a, b).$$

It leads to a contradiction. □

Example 2.3. *:*

(i) If we let $(\mathbb{R}, d)$ be the usual metric space and let $x_n = 1/n$, then (x_n) is a convergent sequence in $\mathbb{R}$.

(ii) If we let $X = (0, 1]$ and d is the metric induced by the usual metric on $\mathbb{R}$, then the sequence $(1/n)$ is divergent in $(0, 1]$. In fact, if $(1/n)$ converges to an element $a \in (0, 1]$, then $\lim 1/n = a$ in $\mathbb{R}$. Then by the uniqueness of limit (see Proposition 2.2), we have $a = 0$. It leads to a contradiction.

Definition 2.4. *Let A be a subset of X. A point $a \in X$ is said to be a limit point of A if for any $r > 0$, we have*

$$(B(a, r) \setminus \{a\}) \cap A \ne \emptyset$$

i.e., for any $r > 0$, there is an element $z \in A$ such that $0 < d(a, z) < r$ (note: $z \ne a$ because $d(a, z) > 0$).
Put $D(A)$ the set of all limit points of A and $\overline{A} = A \cup D(A)$. Also the set $\overline{A}$ is called the closure of A.

Proposition 2.5. *Using the notation above, let $z \in X$. Then the following are equivalent.*

(i) $z \in \overline{A}$.

(ii) $B(z, r) \cap A \ne \emptyset$ for all $r > 0$.

(iii) There is a sequence $(x_n) \in A$ such that $\lim x_n = z$.

Moreover, if A and B are any subsets of X, then we have

(a) $\overline{\emptyset} = \emptyset$.

(b) $\overline{\overline{A}} = \overline{A}$.
(c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Remark 2.6. *(i) In general,* $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
For example, if we consider $X = \mathbb{R}$ *and* $A = (0,1); B = (1,2)$, *then* $A \cap B = \emptyset$ *and* $\overline{A} = [0,1], \overline{B} = [1,2]$. *So, we have* $\emptyset = \overline{A \cap B} \subsetneq \overline{A} \cap \overline{B} = \{1\}$.
(ii) Let $A_1, A_2, ...$ *be an infinite sequence of subsets of* X. *In general,* $\overline{\bigcup_{n=1}^{\infty} A_n} \neq \bigcup_{n=1}^{\infty} \overline{A_n}$.
For example, let $X = \mathbb{R}$ *and* $A_n = [0, 1 - \frac{1}{n})$. *Then* $\overline{\bigcup_{n=1}^{\infty} A_n} = [0,1]$ *but* $\bigcup_{n=1}^{\infty} \overline{A_n} = [0,1)$.

Example 2.7. *(i) Let* $X = \mathbb{R}$ *and* $A = \mathbb{Z}$. *Then* $D(\mathbb{Z}) = \emptyset$ *and* $\overline{A} = \mathbb{Z}$.
(ii) Let $X = \mathbb{R}$ *and* $A = (0,1]$. *Then* $D(A) = [0,1]$ *and* $\overline{A} = [0,1]$.
(iii) Let $X = (0,\infty)$ *and* $A = (0,1]$. *Then* $D(A) = (0,1]$ *and* $\overline{A} = (0,1]$.
(iv) Let $X = \mathbb{R}$ *and* $A = \mathbb{Q}$. *Then* $D(A) = \mathbb{R}$ *and* $\overline{\mathbb{Q}} = \mathbb{R}$ *(A is said to be dense in X if* $\overline{A} = X$*).*
(v) Using the notation as in Example 1.6, we let

$$c_{00} = \{(x_n) \in \ell^\infty : \text{ there are only finitely many } x_n\ 's \neq 0\}.$$

Also c_{00} *is endowed with the* $\|\cdot\|_\infty$.
Then the set c_{00} *is dense in* c_0. *In fact, if* $v = (v_n) \in c_0$, *then for any* $\varepsilon > 0$, *there is* $N \in \mathbb{N}$ *such that* $|v_n| < \varepsilon$ *for all* $n \geq N$. *Now we define* $\xi = (\xi_n)$ *by* $\xi_n = v_n$ *when* $1 \leq n \leq N-1$ *and* $\xi_n = 0$ *when* $n \geq N$. *Then* $\xi \in c_{00}$ *and* $\|v - \xi\|_\infty = \sup_{n \geq N} |v_n| < \varepsilon$. *So* $v \in \overline{c_{00}}$.

Definition 2.8. *A subset A of X is said to be closed in X if* $\overline{A} = A (\Leftrightarrow D(A) \subseteq A)$.

Proposition 2.9. *A subset A of X is closed if and only if for an element* $a \in X$ *having a sequence* (x_n) *in A with* $\lim x_n = a$, *implies* $a \in A$.

Example 2.10. *(i) Let* $X = \mathbb{R}$. *Then* $\mathbb{Z}$ *is a closed subset on* $\mathbb{R}$ *and* $(0,1]$ *is "Not" a closed subset of* $\mathbb{R}$. *However, if* $X = (0,\infty)$, *then* $(0,1]$ *is a closed subset of* $(0,\infty)$.

So, the notion of "Closeness" depends on the choice of X.

(ii) Using the notation as in Examples 1.2 and 2.3, c_0 *is a closed subspace of* ℓ^∞ *and* c_{00} *is not a closed subspace of* ℓ^∞.
Claim : c_0 *is closed in* ℓ^∞.
By Proposition 2.9, we need to show that if $v \in \ell^\infty$ *with a sequence* (ξ_n) *in* c_0 *such that* $\lim_n \|\xi_n - v\|_\infty = 0$, *then* $v \in c_0$.
Now put $v = (v_j)_{j=1}^\infty$ *and* $\xi_n = (\xi_{n,j})_{j=1}^\infty$. *Let* $\varepsilon > 0$. *Since* $\lim_n \|\xi_n - v\|_\infty = 0$, *there is a positive integer N such that* $\|v - \xi_N\|_\infty < \varepsilon$. *This implies that* $|v_j - \xi_{N,j}| < \varepsilon$ *for all* $j \in \mathbb{N}$. *On the other hand, there is a positive integer J such that* $|\xi_{N,j}| < \varepsilon$ *for all* $j \geq J$ *because* $\xi_N \in c_0$. *So, we have*

$$|v_j| < |\xi_{N,j}| + \varepsilon < 2\varepsilon$$

for all $j \geq J$. *Therefore,* $v \in c_0$. *The proof is finished.*

Proposition 2.11. *Using the notation as before, we have the following assertions.*
(i) The whole set X and the empty set $\emptyset$ *both are closed subsets of X.*
(ii) If A and B are the closed subsets of X, then so is $A \cup B$.
(iii) If $(A_i)_{i\in I}$ *is a family of closed subsets of X, then so is the intersection* $\bigcap_{i\in I} A_i$.
(iv) The closure $\overline{A}$ *of A is the smallest closed set containing A, that is,* $\overline{A}$ *is closed and if F is another closed set with* $A \subseteq F$, *then* $\overline{A} \subseteq F$.

Remark 2.12. *The assumption of the finite union of closed sets in Proposition 2.11 (ii) is essential. For example, consider* $X = \mathbb{R}$ *and* $\bigcup_{n=2}^{\infty} [1/n, 1] = (0,1]$.

Exercise 2.13. *Let A be a non-empty subset of X. A point $a \in X$ is called a boundary point of A if $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$ for all $r > 0$, where A^c denotes the complement of A in X. The set of all boundary points, write ∂A, of A is called the boundary of A.*

- *(i) Find the boundaries of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}$.*
- *(ii) Let $X = (0,1) \cup (2,3)$. Find the boundary of the set $(0,1)$ in X.*
- *(iii) Show that the boundary ∂A is a closed subset of X.*
- *(iv) Show that $\overline{A} = A \cup \partial A$.*

Definition 2.14. *A subset V of X is said to be open in X if for each $z \in V$, there is $r > 0$ such that $B(z,r) \subseteq V$.*

Remark 2.15.
- *(i) The notion of open sets depends on the choice of X in which the sets are sitting. For example $(0,1]$ is not open in $\mathbb{R}$ but it is open in the set $(0,1] \cup [2,3]$.*
- *(ii) A subset V of X can be an open and closed subset of X. For example, $(0,1]$ is open and closed subset of $(0,1] \cup [2,3]$.*
- *(iii) A subset V can be neither closed nor open in X. For example, $(0,1]$ is neither closed nor open in $\mathbb{R}$.*

Proposition 2.16. *We have the following assertions.*
- *(i) A subset V is open in X if and only if $X \setminus V$ is closed in X.*
- *(ii) The empty set $\emptyset$ and the whole set X both are open.*
- *(iii) If $\{V_i\}_{i \in I}$ is a family of open subsets of X, then the union $\bigcup_{i \in I} V_i$ is open in X.*
- *(iv) For any finitely many $V_1, ..., V_N$ open subsets of X, we have $V_1 \cap \cdots \cap V_N$ is open in X. For example, $(0,1]$ is neither closed nor open in $\mathbb{R}$.*

Exercise 2.17.
- *(i) Let V be a subset of X. A point $z \in V$ is said to be an interior point of V if there is $r > 0$ such that $B(z,r) \subseteq V$. If we put $int(V)$ the set of all interior points of V, show that $int(V)$ is an open subset of X.*
- *(ii) A metric d on X is said to be non-archimedean if it satisfies the strong triangle inequality, that is, $d(x,y) \leq \max(d(x,z), d(z,y))$ for all x, y and $z \in X$ (see also Example 1.2 (iv)). Show that if d is a non-archimedean metric on X, then for every closed ball $\overline{B}(a,r) := \{x \in X : d(a,x) \leq r\}$ is an open set in X.*

Definition 2.18. *Let $f : X \to Y$ be a function from X into Y. We say that f is continuous at a point $c \in X$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x,y) < \delta$. Furthermore, f is said to be continuous on A if f is continuous at every point in X.*

Definition 2.19. *A bijection $f : X \to Y$ is said to be a homeomorphism if f and its inverse f^{-1} both are continuous. In this case, X is said to be homeomorphic to Y.*

Remark 2.20. *It is clear that f is continuous at $c \in X$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $B(c,\delta) \subseteq f^{-1}(B(f(c),\varepsilon))$.*

Proposition 2.21. *With the notation as above, we have*
- *(i) f is continuous at some $c \in X$ if and only if for any sequence $(x_n) \in X$ with $\lim x_n = c$ implies $\lim f(x_n) = f(c)$.*
- *(ii) The following statements are equivalent.*
  - *(ii.a) f is continuous on X.*
  - *(ii.b) $f^{-1}(W) := \{x \in X : f(x) \in W\}$ is open in X for any open subset W of Y.*
  - *(ii.c) $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is closed in X for any closed subset F of Y.*

Proof. Part (i):
Suppose that f is continuous at c. Let (x_n) be a sequence in X with $\lim x_n = c$. We claim that

$\lim f(x_n) = f(c)$. In fact, let $\varepsilon > 0$, then there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, c) < \delta$. Since $\lim x_n = c$, there is a positive integer N such that $d(x_n, c) < \delta$ for $n \geq N$ and hence $\rho(f(x_n), f(c)) < \varepsilon$ for all $n \geq N$. Thus $\lim f(x_n) = f(c)$.
For the converse, suppose that f is not continuous at c. Then we can find $\varepsilon > 0$ such that for any n, there is $x_n \in X$ with $d(x_n, c) < 1/n$ but $\rho(f(x_n), f(c)) \geq \varepsilon$. So, if f is not continuous at c, then there is a sequence (x_n) in X with $\lim x_n = c$ but $(f(x_n))$ does not converge to $f(c)$. Part $(iia) \Leftrightarrow (iib)$: Suppose that f is continuous on X. Let W be an open subset of Y and $c \in f^{-1}(W)$. Since W is open in Y and $f(c) \in W$, there is $\varepsilon > 0$ such that $B(f(c), \varepsilon) \subseteq W$. Since f is continuous at c, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon)) \subseteq f^{-1}(W)$. So $f^{-1}(W)$ is open in X.
It remains to show that the converse of Part (ii). Let $c \in X$. Let $\varepsilon > 0$. Put $W := B(f(c), \varepsilon)$. Then W is an open subset of Y and thus $c \in f^{-1}(W)$ and $f^{-1}(W)$ is open in X. Therefore, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(W)$. So, f is continuous at c.
Finally, the last equivalent assertion $(ii.b) \Leftrightarrow (ii.c)$ is clearly from the fact that a subset of a metric space is closed if and only if its complement is open in the given metric space (see Proposition 2.16 (i)).
The proof is complete. □

Corollary 2.22. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between metric spaces. Then the composition $g \circ f : X \to Z$ is also continuous on X.*

Proof. It is clear from Proposition 2.21 at once. □

Definition 2.23. *We say that two metrics d_1 and d_2 on a set X are equivalent if there are positive constants c, c' such that $c'd_1(x, y) \leq d_2(x, y) \leq cd_1(x, y)$ for all $x, y \in X$.*

Example 2.24. *Let $X = (0, 1)$ and d be the usual metric on X, that is $d(x, y) := |x - y|$. Define a metric on X by $\rho(x, y) := \frac{|x-y|}{1+|x-y|}$ for $x, y \in (0, 1)$. Then the metrics d and ρ are equivalent on $(0, 1)$. In fact, one can directly check that we have $\rho(x, y) \leq d(x, y) \leq 2\rho(x, y)$ for all $x, y \in (0, 1)$.*

Proposition 2.25. *Let d_1 and d_2 be the metrics on X. If d_1 and d_2 are equivalent, then the identity map $I : (X, d_1) \to (X, d_2)$ is a homeomorphism.*

Proof. It clearly follows from Proposition 2.21. □

3. Complete Metric Spaces

Let (X, d) be a metric space as before.

Definition 3.1. *A sequence (x_n) in X is called a Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.*

Proposition 3.2. *Every convergent sequence is a Cauchy sequence.*

Proof. Let (x_n) be a convergent sequence in X. Suppose that $\lim_n x_n = v \in X$. Then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(v, x_n) < \varepsilon$ for all $n \geq N$. Thus for any $m, n \geq N$, we see that $d(x_m, x_n) \leq d(x_m, v) + d(v, x_n) < 2\varepsilon$. Thus (x_n) is a Cauchy sequence. □

Remark 3.3. *The converse of Proposition 3.2 does not hold in general. For example, if we consider $X = (0, 1]$ and $x_n = 1/n$, then (x_n) is a Cauchy sequence but it is not convergent in $(0, 1]$.*

The following definition is one of important concepts in mathematics world.

Definition 3.4. *X is said to be complete if every Cauchy sequence in X is convergent.*

Remark 3.5. *The completeness of metric spaces are not preserved under homeomorphisms.*
For example, consider $X = \mathbb{R}$. Let $d_1(x, y) := |x - y|$ and $d_2(x, y) := |e^{-x} - e^{-y}|$ for x, y in $\mathbb{R}$. Then the identity map $I : (X,\ d_1) \to (X,\ d_2)$ is a homeomorphism **(check)!** *and $(X,\ d_1)$ is complete. However, $(X,\ d_2)$ is not complete. In fact, if we let $x_n = n$ for $n = 1, 2...$, then (x_n) is Cauchy but not convergent in $\mathbb{R}$ with respect to the metric d_2.*

The following result is a very important motivation of the definition of completeness.

Theorem 3.6. *$\mathbb{R}$ is complete.*

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}$. We first claim that (x_n) must be bounded. Indeed, by the definition of a Cauchy sequence, if we consider $\varepsilon = 1$, then there is a positive integer N such that $|x_m - x_N| < 1$ for all $m \geq N$ and thus we have $|x_m| < 1 + |x_N|$ for all $m \geq N$. So, if we let $M = \max(|x_1|, ..., |x_{N-1}|, |x_N| + 1)$, then we have $|x_n| \leq M$ for all n. Hence (x_n) is bounded.
So, we can now apply the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Let $L := \lim_k x_{n_k}$. We are going to show that $L = \lim_n x_n$.
Let $\varepsilon > 0$. Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq N$. On the other hand, since $\lim_k x_{n_k} = L$, we can find a positive integer K so that $|L - x_{n_k}| < \varepsilon$ for all $k \geq K$. Now if we choose $r \geq K$ such that $n_r \geq N$, then for any $n \geq N$, we have $|x_n - L| \leq |x_n - x_{n_r}| + |x_{n_r} - L| < 2\varepsilon$. Thus (x_n) is convergent with $\lim_n x_n = L$.
The proof is finished. □

Example 3.7. *(i) $\ell^\infty(\mathbb{N}) := \{(x_i)_{i=1}^\infty : \sup_i |x_i| < \infty\}$ is complete under the sup norm $\|\cdot\|_\infty$.*
In fact, notice that if $(\mathbf{x_n})$ is Cauchy sequence in ℓ^∞ and if we let $\mathbf{x_n} = (x_{n,i})_{i=1}^\infty$, then for each $i = 1, 2...$, $(x_{n,i})_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Thus $\lim_n x_{n,i}$ exists in $\mathbb{R}$ for each i. Write $\xi_i := \lim_n x_{n,i} \in \mathbb{R}$ and $\xi := (\xi_i)$. We are now going to show that $\xi \in \ell^\infty$ and $\lim_n \|\xi - x_n\|_\infty = 0$.
Notice that since (x_n) is a Cauchy sequence in ℓ^∞, so, for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - x_m\|_\infty < \varepsilon$ for all $m, n \geq N$ and hence we have

$$|x_{n,i} - x_{m,i}| \leq \sup_k |x_{n,k} - x_{m,k}| = \|\mathbf{x_n} - \mathbf{x_m}\|_\infty < \varepsilon$$

for all $m, n \geq N$ and for all $i = 1, 2....$ So if we fix i and $m \geq N$ and taking $n \to \infty$, then we have $|\xi_i - x_{m,i}| < \varepsilon$ and hence $\|\xi - \mathbf{x_m}\|_\infty < \varepsilon$ for $m \geq N$. From this we see that $\lim_m \|\xi - \mathbf{x_m}\|_\infty = 0$ and thus $\xi \in \ell^\infty$ because ℓ^∞ is a vector space.

(ii) $c_0(\mathbb{N})$ is complete under the sup-norm. In fact every closed subset of a compete metric space must be complete **(why?)**. *Since c_0 is closed in ℓ^∞, c_0 is complete.*

(iii) $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$ all are complete metric spaces under the ℓ^p-norm.

(iv) $C[a,b] := \{f : [a,b] \to \mathbb{R} : f$ is continuous$\}$ is complete under the sup-norm.

Proposition 3.8. *Let (F_n) be a sequence of closed and bounded non-empty subsets of a complete metric space X. For each n, put $diam(F_n) := \sup\{d(x,y) : x, y \in F_n\}$ (the diameter of F_n). Suppose that it satisfies the following conditions.*

(a) $F_1 \supseteq F_2 \supseteq F_3 \cdots\cdots$.

(b) $\lim_n diam(F_n) = 0$.

If X is complete, then there is a unique element $\xi \in X$ such that $\bigcap_n F_n = \{\xi\}$.

Proof. For each F_n, we take an element x_n in F_n. Then by the condition of (a) and (b) above, (x_n) forms a Cauchy sequence in X. Since X is complete, $\xi := \lim x_n$ exists in X. Note that $\xi \in F_n$ for all n because each F_n is closed and $F_m \supseteq F_{m+1} \supseteq \cdots$ for all m. So, $\xi \in \bigcap_n F_n$.
On the other hand, the condition (b) implies that the intersection $\bigcap_n F_n$ contains at most one element. The proof is finished. □

Remark 3.9. *The assumption of completeness of X in Proposition 3.8 is essential. For example, if we consider $X = (0, 1]$ and $F_n = (0, \frac{1}{n+1}]$ for $n = 1, 2...$, then F_n's satisfies the conditions (a) and (b) above but $\bigcap_n F_n = \emptyset$.*

Definition 3.10. *A subset A of X is said to be nowhere dense (or rare) if the closure $\overline{A}$ has no interior points.*

Example 3.11. *Let $X = \mathbb{R}^2$ and $A = \{(x, y) : x + y = 1\}$. Then A is a nowhere dense subset of $\mathbb{R}^2$.*

Exercise 3.12. *Suppose that $\mathbb{R}^n$ is endowed with the usual metric, that is, for $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, $d(x, y) := \sqrt{\sum_{k=1}^n (x_k - y_k)^2}$. Show that if Y is a proper vector subspace of $\mathbb{R}^n$, that is $Y \subsetneq \mathbb{R}^n$, then Y is nowhere dense.*

Exercise 3.13. *Let A be a subset of X.*

(i) Show that if X is complete, then A is complete if and only if A is closed in X.

(ii) Show that if A is complete, then A is closed in X.

Theorem 3.14. *(***Baire Category Theorem***) If X is complete and (A_n) is a sequence of subsets of X with $X = \bigcup A_n$, then there exists A_m which is not nowhere dense, that is $int(\overline{A_m}) \neq \emptyset$.*

Proof. We are going to prove by contradiction. Suppose that all A_n's are nowhere dense sets, that is $int(\overline{A_n}) = \emptyset$ for all $n = 1, 2....$
Fix any point $x_0 \in X$ and $r_0 > 0$. Since $int\overline{(A_1)} = \emptyset$, we have $B(x_0, r_0) \nsubseteq \overline{A_1}$. Thus, there is a point $x_1 \in B(x_0, r_0) \setminus \overline{A_1}$ and $0 < r_1 < r_0/2$ such that $\overline{B(x_1, r_1)} \cap \overline{A_1} = \emptyset$. Similarly, since $B(x_1, r_1) \nsubseteq \overline{A_2}$, there is $x_2 \in B(x_1, r_1)$ and $0 < r_2 < \frac{1}{2} r_1$ such that $\overline{B(x_2, r_2)} \subseteq \overline{B(x_1, r_1)}$ and $\overline{B(x_2, r_2)} \cap \overline{A_2} = \emptyset$. To repeat the same step, we have a sequence (x_n) in X and a sequence of positive numbers (r_n) such that

(i) $B(x_0, r_0) \supseteq \overline{B(x_1, r_1)} \supseteq \cdots \supseteq \overline{B(x_n, r_n)} \supseteq \overline{B(x_{n+1}, r_{n+1})} \supseteq \cdots$;

(ii) $r_n > \frac{1}{2}r_{n+1}$;
(iii) $\overline{B(x_n, r_n)} \cap \overline{A_n} = \emptyset$,

for all $n = 1, 2....$ Then Proposition 3.8 tells us that $\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} = \{\xi\}$ for some point $\xi \in X$. Now since $X = \bigcup A_n$, we have $\xi \in A_m$ for some m. However, we have $\xi \in \overline{B(x_m, r_m)}$ because $\xi \in \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}$ and so, $\xi \in \overline{B(x_m, r_m)} \cap \overline{A_m}$ which contradicts to the condition (iii) above. The proof is finished. □

In fact, if we modify the proof above, we get the following forms of Baire Category Theorem.

Theorem 3.15. Baire Category Theorem: *Let X be a complete metric space. Then we have the following assertions.*

(i) If (V_n) is a sequence of open dense subsets of X , then $\bigcap V_n$ is dense in X.
(ii) If (F_n) is a sequence of nowhere dense closed subsets of X, then $\bigcup_n F_n$ has no interior points.

Proof. We first note that E is a dense subset of X if and only if $X \setminus E$ has no interior points.
For showing the part (i), we need to show that for any $x_0 \in X$ and $r_0 > 0$, we have $B(x_0, r_0) \cap \bigcap V_n \neq \emptyset$. To see this, put $A_n := X \setminus V_n$. Then (A_n) is a sequence of nowhere dense subsets of X. We keep the notation as in the proof of Theorem 3.14 above. Then we have $\xi \in B(x_0, r_1)$ and $\xi \notin A_n$ for all n. This implies that $\xi \in B(x_0, r_0) \cap \bigcap V_n$ and hence $B(x_0, r_0) \cap \bigcap V_n \neq \emptyset$ as desired.
For the part (ii), let $V_n := X \setminus F_n$. Then (V_n) be a sequence of open dense subsets of X. Part (i) implies that $\bigcap V_n$ is dense in X and so $\bigcup F_n = (\bigcap V_n)^c$ has no interior points. The proof is complete. □

Definition 3.16. *A subset of a metric space X is called a* G_δ**-set** *if it is a countable union of open subsets of X.*

Proposition 3.17. *If $f : X \to \mathbb{R}$ is a bounded function defined on a metric space, then the set C of all continuous points of f, that is $C := \{x \in X : f \text{ is continuous at } x\}$, is a G_δ set.*

Proof. We first recall some usual notation. For each $c \in X$, put

$$\overline{f}(c) := \overline{\lim}_{x \to c} f(x) := \inf_{r>0} \sup_{x \in B(c,r)} f(x) \quad \text{and} \quad \underline{f}(c) := \underline{\lim}_{x \to c} f(x) := \sup_{r>0} \inf_{x \in B(c,r)} f(x).$$

Then $\underline{f}(c) \leq \overline{f}(c)$ for all $c \in X$ and f is continuous at c if and only if $\underline{f}(c) = \overline{f}(c)$. Thus, we have

$$C = \bigcap_n \{c \in X : \overline{f}(c) - \underline{f}(c) < \frac{1}{n}\}.$$

Let $G_n := \{c \in X : \overline{f}(c) - \underline{f}(c) < \frac{1}{n}\}$. We are going to show that each G_n is an open subset of X for all $n = 1, 2....$ For convenience, let $\beta(c, r) : \sup_{y \in B(c,r)} f(y)$ and $\alpha(c, r) := \inf_{y \in B(c,r)} f(y)$ for $r > 0$ and $c \in X$. Then we have $\alpha(c, r) \leq \underline{f}(c) \leq \overline{f}(c) \leq \beta(c, r)$ and $\alpha(c, r) \uparrow \underline{f}(c)$ and $\beta(c, r) \downarrow \overline{f}(c)$ as $r \to 0+$. Thus, we have

$$0 \leq \beta(c, r) - \alpha(c, r) \leq \overline{f}(c) - \underline{f}(c) \text{ for all } r > 0$$

and

$$\beta(c, r) - \alpha(c, r) \to \overline{f}(c) - \underline{f}(c) \quad \text{as } r \to 0+ .$$

Thus, if we fix n and $c \in G_n$, then there is $r_0 > 0$ such that $\beta(c, r) - \alpha(c, r) < \frac{1}{n}$ whenever $0 < r < r_0$. Notice that if $x_1 \in B(c, r/2)$, then $B(x_1, r/2) \subseteq B(c, r)$. This gives

$$\alpha(c, r) \leq \alpha(x_1, r/2) \leq \underline{f}(x_1) \leq \overline{f}(x_1) \leq \beta(x_1, r/2) \leq \beta(c, r) \quad \text{for all } r > 0.$$

Therefore, if we fix $0 < r < r_0$, then we have $x_1 \in G_n$ whenever $x_1 \in B(c, r/2)$ and thus, $B(c, r/2) \subseteq G_n$. The proof is finished. □

Example 3.18. *The set of all rational numbers $\mathbb{Q}$ is "Not" a G_δ-set.*

Proof. Suppose not. Let (G_n) be a sequence of open subsets of $\mathbb{R}$ such that $\mathbb{Q} = \bigcup G_n$. Then each G_n is an open dense subset of $\mathbb{R}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. On the other hand, since $\mathbb{Q}$ is a countable set, we can write $\mathbb{Q} = \{r_1, r_2, ...\}$ as a sequence set. Now for each n, put $V_n := G_n \setminus \{r_n\}$. Then V_n is still an open dense subset of $\mathbb{R}$. However, $\bigcap V_n = \emptyset$ because $\mathbb{Q} = \bigcup G_n$. It contradicts to the Baire's Theorem 3.15 (i) above. $\square$

Example 3.19. *There is $f \in C[0,1]$ such that $f'(x)$ does not exist for all $x \in (0,1)$.*

Proof. Recall that for each $f, g \in C[0,1]$, put $d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$. Then $(C([0,1], d)$ is a complete metric space.
Now for each $n = 1, 2....$, put

$$A_n := \{f \in C[0,1] : \text{ there is } x_0 \in [0,1] \text{ such that } |f(x) - f(x_0)| \le n|x - x_0| \text{ for all } x \in [0,1]\}.$$

The proof is divided by he following claims.
Claim 1: Each A_n is a non-empty closed subset of $C[0,1]$.
Notice that each $A_n \neq \emptyset$ since A_n contains all constant functions. Now fix n. Let (f_i) be a sequence in A_n with $f := \lim_i f_i$ in $C[0,1]$, that is, $d(f_i, f) \to 0$. Since $f_i \in A_n$, there is $x_i \in [a,b]$ such that $|f_(x) - f_(x_i)| \le n|x - x_i|$ for all $i = 1, 2...$ Notice that by the Bolzano-Weierstrass Theorem, (x_i) has a convergent subsequence (x_{i_k}). Let $c := \lim_k x_{i_k} \in [0,1]$. Now let $\varepsilon > 0$. Since $d(f_i, f) \to 0$ as $i \to \infty$, we have $d(f_{i_k}, f) \to 0$ as $k \to \infty$. Thus, there a positive integr K such that $|f_{i_k}(x) - f(x)| < \varepsilon$ for all $k \ge K$ and for all $x \in [0,1]$. Then we have

$$\begin{aligned} |f(x) - f(x_{i_k})| &\le |f(x) - f_{i_k}(x)| + |f_{i_k}(x) - f_{i_k}(x_{i_k})| + |f_{i_k}(x_{i_k}) - f(x_{i_k})| \\ &\le \epsilon + n|x - x_{i_k}| + \varepsilon \quad \text{ for all } k \ge K \text{ and for all } x \in [0,1]. \end{aligned}$$

Now taking $k \to \infty$, then we have

$$|f(x) - f(c)| \le 2\varepsilon + n|x - c|$$

for all $\varepsilon > 0$ and for all $x \in [0,1]$. Hence we have $|f(x) - f(c)| \le n|x - c|$ for all $x \in [0,1]$ and so $f \in A_n$ as desired.
Claim 2: $int(A_n) = \emptyset$ for all $n = 1, 2....$
Fix a positive integer n. We want to show that for any $f \in C[0,1]$ and $r > 0$, there is $g \in C[0,1]$ with $d(f,g) < r$ but $g \notin A_n$.
To see this, one can choose a piecewise linear function g with $d(f,g) < r$ and such that each segment of slope greater than $n + 1$. Then $g \notin A_n$ as required.
Since $C[0,1]$ is complete, the Baire Theorem tells us that $\bigcup A_n \subsetneq C[0,1]$. Therefore, there is $h \in C[0,1] \setminus \bigcup A_n$. The proof is finished after the following claim.
Claim 3: $h'(x)$ does not exist for all $x \in (0,1)$.
To see this, suppose that $h'(x_0)$ exist for some $x_0 \in (a,b)$. Then by the definition of derivative, there is $r > 0$ such that $|h(x) - h(x_0)|(1 + |h'(x_0)|)|x - x_0|$ for all $x \in (x_0 - r, x_0 + r) \subseteq [0,1]$. Notice that since the function $|h(x) - h(x_0)|/|x - x_0|$ is continuous on $[0, x_0 - r] \cup [x_0 + r, 1]$, the function $|h(x) - h(x_0)|/|x - x_0|$ is bounded by some $M > 0$ on $[0, x_0 - r] \cup [x_0 + r, 1]$. Now choose $N \in \mathbb{N}$ such that $N \ge \max(M, |h'(x_0)| + 1)$. Thus, we have $|h(x) - h(x_0)| \le N|x - x_0|$ for all $x \; in[0,1]$ and so $h \in A_N$ which contradicts to $h \notin A_n$ for all $n \in \mathbb{N}$. $\square$

Exercise 3.20. *Let X be a complete metric space and let $\mathcal{F}$ be a collection of continuous real valued functions defined on X. Suppose that we have*

$$\sup\{|f(x)| : f \in \mathcal{F}\} < \infty \quad \textit{for all } x \in X.$$

Then there is a non-empty open subset U of X such that $\mathcal{F}$ is uniformly bounded on U, that is, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in U$ and for all $f \in \mathcal{F}$.
(Hint: for each $n \in \mathbb{N}$, consider the set $A_n := \{x \in X : |f(x)| \leq n, \quad \text{for all } f \in \mathcal{F}\}$).

Theorem 3.21. Banach fixed point theorem: *Let $T : X \to X$ be a contraction, that is, there is $0 < r < 1$ such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. If X is complete, then there exists a unique fixed point $\xi \in X$ for T, that is $T(\xi) = \xi$.*

Proof. Clearly, T is continuous on X.
Existence: Fix any point $x_1 \in X$. Put $x_{n+1} = Tx_n$ for $n = 1, 2, ...$ Notice that we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq rd(x_n, x_{n-1})$$

for all $n = 1, 2, ...$ This implies that $d(x_{n+1}, x_n) \leq r^{n-1}d(x_n, x_{n-1})$ for $n = 2, 3, ...$ This implies that for $n < m$, we have

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} r^{k-1}d(x_2, x_1).$$

This implies that (x_n) is a Cauchy sequence in X since $0 < r < 1$. Hence $\xi = \lim x_n$ exists in X and so we have $\xi = T\xi$ by taking $n \to \infty$ in the equation $x_{n+1} = Tx_n$.
Uniqueness: assume that ξ and ξ' are the fixed points for T, that is $T\xi = \xi$ and $T\xi' = \xi'$. Then we have $d(\xi, \xi') = d(T^n\xi, T^n\xi') \leq r^nd(\xi, \xi')$ for all $n = 1, 2....$ Since $0 < r < 1$, we have $d(\xi, \xi') = 0$ and so $\xi = \xi'$ as desired. □

Proposition 3.22. *Let (X, d) be a metric space. Then there is a metric space (X_0, d_0), together with a linear map $i : X \to X_0$, satisfies the following conditions.*

(i) X_0 is complete.
(ii) The map i is an isometry, that is, $d_0(i(x), i(y)) = d(x, y)$ for all $x, y \in X$.
(iii) the image $i(X)$ is dense in X_0, that is, $\overline{i(X)} = X_0$.

Moreover, such pair (X_0, d_0) is unique up to isometric isomorphism in the following sense.
If (W, d_1) is complete and there is an isometry $j : X \to W$ such that $\overline{j(X)} = W$, then there is an isometric isomorphism ψ from X_0 onto W such that

$$j = \psi \circ i : X \to X_0 \to W.$$

In this case, the pair (X_0, i) is called the completion of X.

Proof. We give the outline proof here. The proof is divided by several steps.
Let $\mathcal{C}$ be the collection of all Cauchy sequences in X. We define a relation on $\mathcal{C}$ as follows. For (x_n) and (y_n) in $\mathcal{C}$, we say that $(x_n) \sim (y_n)$ if $\lim d(x_n, y_n) = 0$. Clearly, $'' \sim ''$ is an equivalence relation on $\mathcal{C}$. For each $(x_n) \in \mathcal{C}$, put $[(x_n)]$ the equivalence class of (x_n). Put $\widetilde{X} := \{[(x_n)] : (x_n) \in \mathcal{C}\}$. Define

$$\widetilde{d} : ([(x_n)], [(y_n)]) \in \widetilde{X} \times \widetilde{X} \mapsto \lim_n d(x_n, y_n) \in \mathbb{R}_+.$$

Claim 1: $\widetilde{d}$ is a well defined function on $\widetilde{X} \times \widetilde{X}$, that is $\lim_n d(x_n, y_n)$ exists and does not depend on the choice of (x_n) and (y_n) in $[(x_n)]$ and $[(y_n)]$ respectively.
Claim 2: $(\widetilde{X}, \widetilde{d})$ is a metric.
Define a mapping $Q : X \to \widetilde{X}$ by $Q(x) := (x_n)$ for $x \in X$ and where $x_n = x$ for all $n = 1, 2....$
Claim 3: Q is an isometry.
Claim 4: $\overline{Q(X)} = \widetilde{X}$.
The result will then follow the final claim.
Claim 5: $(\widetilde{X}, \widetilde{d})$ is complete. (**Hint:** Let (ξ_i) be a Cauchy sequence in $(\widetilde{X}, \widetilde{d})$. Then by Claim 4, for

each positive integer i, there is $z_i \in X$ such that $\widetilde{d}(Q(z_i), \xi_i) < 1/i$. One can show that $z := (z_i)$ is a Cauchy in X and $\widetilde{d}(\xi_i, [z]) \to 0$ as $i \to \infty$ and hence $(\widetilde{X}, \widetilde{d})$ is complete as desired. $\square$

Example 3.23. Proposition 3.22 cannot give an explicit form of the completion of a given metric space. The following examples are basically due to the uniqueness of the completion.

(*i*) The completion of a complete metric space is itself.

(ii) The completion of $\mathbb{Q}$ is $\mathbb{R}$.

(*ii*) The completion of the finite sequence space c_{00} under the sup-norm is the null sequence space c_0.

(*iii*) The completion of $C_c(\mathbb{R})$ is $C_0(\mathbb{R})$.

4. Topological spaces

Definition 4.1. *Let X be a set. A collection $\mathfrak{T}$ of subsets of X is called a topology on X if it satisfies the following conditions:*

(i) the empty set $\phi \in \mathfrak{T}$ and $X \in \mathfrak{T}$,
(ii) whenever $\{U_i\}_{i\in I}$ is a sub-collection of $\mathfrak{T}$, implies that the union $\bigcup_{i\in I} U_i \in \mathfrak{T}$,
(iii) whenever $\{U_1, ..., U_N\}$ is a finite sub-collection of $\mathfrak{T}$, implies that the intersection $U_1 \cap \cdots \cap U_N \in \mathfrak{T}$.

In this case, the pair $(X, \mathfrak{T})$ is called a topological space. Also, each element in $\mathfrak{T}$ is called an open subset of X.

Example 4.2. *Let X be a set. The following collections form the topologies on X.*

(i) Let $\mathfrak{T}_0 := \{\emptyset, X\}$. Clearly, $\mathfrak{T}_0$ is a topology and $\mathfrak{T}_0$ is called the trivial topology on X.
(ii) Let $\mathfrak{T}_d$ be the collection of all subsets of X, that is the power set of X. In this case, $\mathfrak{T}_d$ is called the discrete topology on X.
Notice that $\mathfrak{T}$ is the discrete topology if and only if every singleton set $\{x\} \in \mathfrak{T}$ for all $x \in X$.
(iii) Let (X, ρ) be a metric space. Recall that a subset U is called an open subset of X if for every $x_0 \in U$, there is $r > 0$ such that the open ball $B(x_0, r) := \{x \in X : \rho(x_0, x) < r\} \subseteq U$, that is every point in U is an interior point. If we let $\mathfrak{T}_\rho$ be the collection of all open subsets of X in this sense, then the collection $\mathfrak{T}_\rho$ is a topology on X. Thus, a metric space (X, ρ) is a topological space and $\mathfrak{T}_\rho$ is called the metric topology induced by the metric ρ.
(iv) Assume that X is an infinite set. Let $\mathfrak{T} := \{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite }\}$. Then $\mathfrak{T}$ is a topology on X. This topology is called the co-finite topology on X.

Remark 4.3. *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the topologies on X. We say that the topology $\mathfrak{T}_2$ is stronger then the topology $\mathfrak{T}_1$ if $\mathfrak{T}_2 \supseteq \mathfrak{T}_1$.*
Thus the trivial topology $\mathfrak{T}_0$ and the discrete topology $\mathfrak{T}_d$ are the weakest topology and the strongest topology on X respectively.

The following notation is very important in mathematics.

Definition 4.4. *A topological space $(X, \mathfrak{T})$ is called a Hausdorff space or T_2 space if whenever for any pair of elements $x, y \in X$ with $x \neq y$, there are disjoint open neighbourhoods U and V of x and y respectively, that is U and V are open sets such that $x \in U; y \in V$ and $U \cap V = \emptyset$.*

Example 4.5. *Clearly, all metric topologies and discrete topology are Hausdorff.*
The co-finite topology defined in example 4.2 (iv) is Not Hausdorff. In fact, for each pair of open subsets U and V, we have $U \cap V \neq \emptyset$. Hence it is not a metric topology.

Remark 4.6. *As in the metric spaces case, it is naturally to define the notation about convergent sequences in general topological spaces. We say that a sequence (x_n) in a topological space X is convergent if there is an element $c \in X$ such that for every open neighbourhood V of c, there is n_0 such that $x_n \in V$ for all $n \geq n_0$. In this case, c is called a limit of (x_n).*
If $(X, \mathfrak{T})$ is Hausdorff, one can show the uniqueness of limit as in the metric spaces case provided its limit exists. However, the following example shows that it is not the case for non-Hausdorff spaces.

Example 4.7. *Let $\mathcal{T}$ be the co-final topology defined on $\mathbb{R}$, that is $V \in \mathcal{T}$ if and only if the complement V^c is finite. Let $x_n = n$ for all $n = 1, 2....$ Then (x_n) converges to every point in $\mathbb{R}$. To see this, let c be any point in $\mathbb{R}$ and let V be any open neighbourhood of c. Then by the definition of the co-finite topology, the complement V^c is finite. Thus, there is $n_0 \in \mathbb{N}$ such that $n_0 > t$ for all $t \in V^c$. This implies that $x_n \in V$ for all $n \geq n_0$ and so, c is a limit of (x_n) for all $c \in \mathbb{R}$.*

Definition 4.8. *Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Put*

$$\mathcal{T}_A := \{V \cap A : V \in \mathcal{T}\}.$$

One can directly check that the collection $\mathcal{T}_A$ forms a topology on A, that is the pair $(A, \mathcal{T}_A)$ becomes a topological space. In this case, $\mathcal{T}_A$ is called the relative topology on A. In addition, those elements in $\mathcal{T}_A$ are said to be open in A.

Remark 4.9. *We keep the notation as in Definition 4.8 above. In general, a subset E is open in A that it may not be open in X. For example, if $\mathbb{R}$ is endowed with the metric topology, then the set $[0, 1/2)$ is open in $[0, 1]$ but it is not open in $\mathbb{R}$.*
In fact, it is easy to see that every open subset of A is also open in X if and only if A is open in X.

Definition 4.10. *We keep the notation as before. Let X be a topological space and let A be a subset of X.*

- *(i) A subset F of X is called a closed set if the complement F^c is open.*
- *(ii) A subset E of A is said to be closed in A if $A \setminus E \in \mathcal{T}_A$, that is E is closed in A with respect to the relative topology.*
- *(iii) A point $x_0 \in X$ is called a cluster point (or limit point of A if whenever V is an open neighbourhood V of x_0, we have $A \cap (V \setminus \{x_0\}) \neq \emptyset$. We write $D(A)$ for the set of all cluster points of A.*
- *(iv) The set $\overline{A} := A \cup D(A)$ is called the closure of A.*

The following properties of closed sets can be directly shown by the definition.

Proposition 4.11. *We keep the notation as before. Let A be a subset of a topological space X. Then we have the following assertions.*

- *(i) $\emptyset$ and X are closed sets.*
- *(ii) whenever $\{F_i\}_{i \in I}$ is a collection of closed sets, implies that the intersection $\bigcap_{i \in I} F_i$ is also closed.*
- *(iii) whenever $\{F_1, ..., F_N\}$ is a finite collection of closed sets, implies that the union $F_1 \cup \cdots \cup F_N$ is also closed.*
- *(iv) an element $x_0 \in \overline{A}$ if and only if $V \cap A \neq \emptyset$ for any open neighbourhood V of x_0.*
- *(v) $\overline{A}$ is the smallest closed set containing A, that is, if F is a closed set containing A, then $\overline{A} \subseteq F$.*
- *(vi) A is a closed subset of X if and only if $D(A) \subseteq A$ if and only $\overline{A} = A$.*
- *(vii) $\overline{\overline{A}} = \overline{A}$.*

Definition 4.12. *Let $(X, \mathcal{T})$ a be topological space.*

- *(i) A collection of open subsets, say $\mathcal{B}$, is said to be an open base for $\mathcal{T}$ if for every $x_0 \in X$ and for every open neighbourhood V of x_0, there is an element $B \in \mathcal{B}$ such that $x_0 \in B \subseteq V$. Furthermore, if X has a countable open base, then X is said to be second countable.*
- *(ii) A collection of open subsets $\mathcal{S}$ is called an open subbase if the collection $\{S_1 \cap \cdots \cap S_n : S_1, ..., S_n \in \mathcal{S}; n = 1, 2...\}$ forms an open base.*

Remark 4.13. *Clearly, we see that a collection $\mathbb{B}$ of open sets forms an open base if and only of every open set V is the union of some elements in $\mathcal{B}$.*

Example 4.14. *(i) The collection of all open balls in a metric space forms an open base for the metric topology.*
(ii) The collection $\{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ forms an open subbase for the metric topology on $\mathbb{R}$.

Exercise 4.15. *A topological space is said to be separable if there is a countable subset. Show that every separable metric space is second countable.*
However, the following example shows that this is not the case for general topological spaces.

Example 4.16. *Let $X = \mathbb{R}$ and let $\mathcal{T}_{cf}$ be the co-finite topology defined as in Example 4.2. Notice that every infinite subset of $\mathbb{R}$ is dense in $\mathbb{R}$ in the co-finite topology (**Why?**). Hence $\mathbb{R}$ is separable in this topology. However, it is not second countable. To see this, suppose not. Let $\mathcal{B}$ be a countable open base for $\mathcal{T}_{cf}$. Fix $x_0 \in \mathbb{R}$. If we let $\mathcal{B}(x_0) := \{B \in \mathcal{B} : x_0 \in B\}$, then $\bigcap_{B \in \mathcal{B}(x_0)} B = \{x_0\}$. To see this, if there is $x_1 \in B$ for every $B \in \mathcal{B}(x_0)$ with $x_0 \neq x_1$, then $x_0 \in W := \mathbb{R} \setminus \{x_1\} \in \mathcal{T}_{cf}$. Thus, there is $B_1 \in \mathcal{B}$ such that $x_0 \in B_1 \subseteq W$. Hence, $x_1 \in B_1 \subseteq W$ which leads to a contradiction.*
From this we have $\mathbb{R} \setminus \{x_0\} = \bigcup_{B \in \mathcal{B}} B^c$. It leads to a contradiction because $\mathcal{B}$ is countable and each B^c is finite.

Proposition 4.17. *Let $\mathcal{S}$ be a collection of subsets of X. Then there is the smallest topology $\mathcal{T}$ containing $\mathcal{S}$. The smallest in here in the sense that if $\mathcal{T}_1$ is another topology containing $\mathcal{S}$, then $\mathcal{T} \subseteq \mathcal{T}_1$. In this case, we say that the topology $\mathcal{T}$ is generated by $\mathcal{S}$.*

Proof. If we let $\mathfrak{X}$ be the collection of topologies on X that contains $\mathcal{S}$ and put

$$\mathcal{T} = \bigcap_{\mathcal{J} \in \mathfrak{X}} \mathcal{J},$$

then $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$.
Alternatively, the topology $\mathcal{T}$ can be explicitly constructed as follows.
Let $\mathcal{C} := \{S_1 \cap \cdots \cap S_n : S_1, ..., , S_n \in \mathcal{S}; n = 1, 2...\}$. Then the collection of the unions of the members in $\mathcal{C}$, that is $V = \bigcup C_i$, where $\{C_i\} \subseteq \mathcal{C}$. Then this collection is the smallest topology containing $\mathcal{S}$ as desired. $\square$

Definition 4.18. *Let $f : X \to Y$ be a mapping between the topological spaces X and Y. We say that f is continuous at a point x_0 in X if whenever V is an open neighbourhood of $f(x_0)$, there exists an open neighbourhood U of x_0 such that $x_0 \in U \subseteq f^{-1}(V) := \{x \in X : f(x) \in V\}$.*
If f is continuous at every point in X, then f is said to be a continuous map.
Furthermore, f is said to be a homeomorphism if f is a continuous bijection and the inverse $f^{-1} : Y \to X$ is also continuous. In this case, we say that X and Y are homeomorphic. We write $X \cong Y$.

Remark 4.19. Warning!! *Recall a fact about metric spaces that*

for a mapping $f : X \to Y$ between the metric spaces X and Y,
f is continuous at $c \in X$ if and only if whenever a sequence (x_n) converges to c, then the sequence $(f(x_n))$ converges to $f(c)$.

The following example shows that the above statement does not hold in general topological spaces.

Example 4.20. *Let $\mathcal{T}_{cc}$ be the co-countable topology defined on $\mathbb{R}$, that is $V \in \mathcal{T}_{cc}$ if and only if $\mathbb{R} \setminus V$ is countable.* **(Check this is a topology!)***.*
*Let $\mathcal{T}_u$ be the usual metric topology on $\mathbb{R}$. Then the identity map $id : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_u)$ is sequentially continuous, but it is not continuous (***Why ?***).*
In here f is said to be sequentially continuous if for every $c \in X$, whenever a sequence (x_n) converges to c, then the sequence $(f(x_n))$ converges to $f(c)$,

However, one can use the same argument as in the metric spaces to show the following result.

Proposition 4.21. *Let X and Y be topological spaces. Let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) f is continuous.
(ii) $f^{-1}(V)$ is an open subset of X for any open subset V of Y.
(iii) $f^{-1}(F)$ is a closed subset of X whenever F is a closed subset of Y.

Proposition 4.22. *Let Ω be a non-empty set. Let $\{X_i : i \in I\}$ be a family of topological spaces and $\mathcal{F}$ be a family of functions $f_i : \Omega \to X_i$, $i \in I$. Then there is the weakest topology $\mathcal{T}$ on Ω such that each $f_i \in \mathcal{F}$ is continuous.*

Proof. Put

$$\mathcal{S} := \{f_i^{-1}(V_i) : \ V_i \text{ is open subset of } X_i,\ i \in I\}.$$

Then by Propositions 4.17 and 4.21, the result follows. □

Definition 4.23. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Consider the product set*

$$X := \prod_{i \in I} X_i := \{f : I \to \bigcup_{i \in I} X_i : f(i) \in X_i \textit{for all } i \in I\}.$$

For each $i \in I$, let $p_i : f \in X \mapsto f(i) \in X_i$ be the natural projection. Then the topology genearted by the family $\{p_i : i \in I\}$ is called the product topology on X.

Exercise 4.24. *To retain the notation as in Definition 4.23, the product topology on $X := \prod_{i \in I} X_i$ is Hausdorff if and only if each X_i is Hausforff space.*

Definition 4.25. *Let X be a topological space. Let " $\sim$ " be an equivalence relation on X. Write $X/\sim$ (called* **quotient space** *) for the set of all equivalence classes on X under the relation $\sim$. Let $\pi : X \to X/\sim$ be the natural projection, that is $\pi(x) = \bar{x} := \{x' \in X : x \sim x'\}$. Then there is the strongest topology on $X/\sim$ such that π is continuous. We call the* **quotient topology** *about this.*

Exercise 4.26. *We keep the notation as in Definition 4.25. Then the quotient topology on $X/\sim$ is given by the collection*

$$\{\pi^{-1}(V) : \ V \textit{ is an open subset of } X \ \}.$$

Example 4.27. *Let $X = [0, 1]$. Assume that $0 \sim 1$. Then the quotient space $[0, 1]/\sim$ is homeomorphic to the unit circle $\{(x, y) : x^2 + y^2 = 1\}$.*

Remark 4.28. *The following shows that the quotient space of a Hausdorff space may "NOT be Hausdorff.*

Example 4.29. *For $x, y \in \mathbb{R}$, we say that $x \sim y$ if $x - y \in \mathbb{Z}$. Then the quotient space $\mathbb{R}/\mathbb{Z}$ is Hausdorff in the quotent topology.*

Exercise 4.30. *For $x, y \in \mathbb{R}$, we say that $x \sim y$ if $x - y \in \mathbb{Q}$. Show that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Haudorff.*

5. Compact spaces

From now on, an open cover of a topological space X means that it is a collection of open subsets of X, say $\{U_i\}_{i\in I}$ such that $X = \bigcup_{i\in I} U_i$. We now come to a very important notation in mathematics.

Definition 5.1. *A topological space X is said to be compact if whenever $\mathcal{U} := \{U_i\}$ is an open cover of X, then there are finitely many elements in $\mathcal{U}$, say $U_{i_1}, ..., U_{i_N}$, such that $X = U_{i_1} \cup \cdots \cup U_{i_N}$.*
In particular, a subset A of X is said to be compact when it is compact in the relative topology. In this case, if for any family of open subsets $\{U_i\}_{i\in I}$ of X with $A \subseteq \bigcup_{i\in I} U_i$, then there are finitely many $U_{i_1}, ..., U_{i_N}$ such that $A \subseteq U_{i_1} \cup \cdots \cup U_{i_N}$.

Example 5.2.
- *(i) : Every closed and bounded subinterval is a compact set under the metric topology due to the nested interval theorem.*
- *(ii) $(0, 1)$ is not a compact set under the metric topology but it is compact under the co-final topology* **Why?**.
- *(iii) If $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$ is endowed with the usual metric topology, that is $\rho(x, y) := \sqrt{\sum_{k=1}^\infty (x_k - y_k)^2}$ for $x = (x_1, x_2, ..), y = (y_1, y_2, ..) \in \mathbb{R}|^\infty$, then the closed unit ball $B := \{x \in \mathbb{R}^\infty : \sum x_k^2 \leq 1\}$ is not compact under the metric topology. (Note that B is a closed and bounded set under the metric ρ).*

The following assertions can be directly shown by the definition of the compactness.

Proposition 5.3. *Let X, Y be topological spaces and $A \subseteq X$. We have the following assertions.*
- *(i) If X is compact and A is closed, then A is compact.*
- *(ii) If X is Hausdorff and A is compact, then A is closed.*
- *(iii) Let $f : X \to Y$ be a continuous map. If X is compact, then so is $f(X)$. Further, if f is a continuous bijection and Y is Hausdorff, then f is a homeomorphism.*

Exercise 5.4. *Let (X, ρ) be a compact metric space, that is X is compact under the metric topology given by ρ.*
If $f : X \to \mathbb{R}$ is a continuous map, then f is uniform continuous on X, that is for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in X$ with $\rho(x, y) < \delta$.

Definition 5.5. *A topological space X is a said to be sequentially compact if for every sequence in X has a convergent subsequence.*

Example 5.6.
- *(i) Every closed and bounded interval in $\mathbb{R}$ is sequentially compact in the metric topology due to the Bolzano-Weierstrass Theorem: every bounded sequence in $\mathbb{R}$ has a convergent subsequence. However, every open interval is not sequentially compact.*
- *(ii) $\mathbb{R}$ is sequentially compact under the co-final topology $\mathfrak{T}_{cf}$. To see this, let (x_n) be a sequence in $\mathbb{R}$. If $\{x_n : n = 1, 2..\}$ is a finite set, then clearly (x_n) has a convergent subsequence. Suppose that the set $\{x_n : n = 1, 2..\}$ is infinite. Let c be any point in $\mathbb{R}$. Then by the definition of the co-finite topology, we see that $x_n \to c$ because $\{x_n\}$ is infinite.*
- *(iii) The closed unit ball B defined as in Example 5.2(iii) is not sequentially compact since for the natural basis (e_n), we always have $\rho(e_n, e_m) = \sqrt{2}$ for $n \neq m$.*

Remark 5.7. *For general topological spaces, "sequentially compactness" $\not\Leftrightarrow$ "compactness".*
However, for metric spaces these two notations are the same.

Proposition 5.8. *Let X be a metric space. If X is a compact, then it is sequentially compact.*

Proof. Suppose that X is not sequentially compact. Then there is a sequence (x_n) in X which has no convergent subsequences. Let $A := \{x_n : n = 1, 2...\}$. If A is finite, then (x_n) clearly has a convergent subsequence that contradicts to the assumption of (x_n). Hence, the set A is infinite. Since (x_n) has no convergent subsequence, then for every point $x \in X$ there is $r_x > 0$ such that the open ball $B(x, r_x) \cap A$ at most one point. Clearly, we have $X = \bigcup_{x\in X} B(x, r_x)$. Then by the compactness of X, there are finitely many elements, $x_1, ..., x_N$ in X such that $X = B(x_1, r_{x_1}) \cup \cdots \cup B(x_N, r_{x_N})$. However, $A \cap B(x, r_x)$ is finite for all $x \in X$, so A is finite. It leads to a contradiction. □

We are going to show the converse of Proposition 5.8 for metric spaces. The proof is divided by several results.

Lemma 5.9. *Let (X, d) be a compact metric space. If $\mathcal{U}$ is an open cover of X, then there is $\lambda > 0$ such that for every $x \in X$, there is $G \in \mathcal{U}$ such that $B(x, \lambda) \subseteq G$. (in this case such λ is called a Lebesgue number for the covering $\mathcal{U}$.)*

Proof. For each subset A of X, let $diam(A) := \sup\{d(x, y) : x, y \in A\}$ denote the diameter of A. Put

$$\mathcal{A} := \{A \subseteq X : A \subsetneqq G \quad \text{for all } G \in \mathcal{U}\}.$$

Note that if the collection $\mathcal{A} = \emptyset$, then we can take any $\lambda > 0$ so that the result follows. Now suppose that $\mathcal{A} \neq \emptyset$. Set

$$R := \inf\{diam(A) : A \in \mathcal{A}\}.$$

Notice that if $R > 0$, then we take $\lambda := \frac{1}{4}R > 0$ as desired.
Hence, it suffices to show that $R > 0$. Suppose that $R = 0$. Then there is a sequence (A_n) in $\mathcal{A}$ such that $diam(A_n) < 1/n$ for all $n = 1, 2....$ Now fix any element $x_n \in A_n$. Then by the assumption of X being sequentially compact, (x_n) has a convergent subsequence (x_{n_k}). Let $z := \lim_k x_{n_k}$. Since $\mathcal{U}$ is an open cover of X, then $z \in B(z, r) \subseteq G$ for some $G \in \mathcal{U}$ and for some $r > 0$. On the other hand, we can choose K large enough so that $\frac{1}{n_K} < r$ and $x_{n_K} \in B(z, \frac{1}{2}r)$ because $x_{n_k} \to z$ as $k \to \infty$. This implies that for any element $y \in A_{n_K}$, we have $d(z, y) \leq d(z, x_{n_K}) + d(x_{n_K}, y) \leq \frac{1}{2}r + \frac{1}{n_K} < r$. Thus, we have $A_{n_K} \subseteq B(z, r) \subseteq G$ that leads to a contradiction since $A_{n_K} \in \mathcal{A}$. The proof is complete. □

Proposition 5.10. *If (X, d) is a compact metric space, then for any $r > 0$, there is finitely many elements $x_1, .., x_N$ in X such that $X = \bigcup_{k=1}^N B(x_k, r)$.*

Proof. Suppose not. Then there is $r > 0$ without such property. Fix any $x_1 \in X$. Then by the assumption $B(x_1, r) \subsetneq X$. Hence, there is $x_2 \in X \setminus B(x_1, r)$. By using the assumption again, there is $x_3 \in X \setminus \bigcup_{k=1}^2 B(x_k, r)$. To repeat the same step, we have a sequence in (x_n) in X such that $x_{n+1} \in \bigcup_{k=1}^n B(x_k, r)$ for all $n = 1, 2....$ Thus, we have $d(x_m, x_n) \geq r$ for all $m \neq n$. This implies that (x_n) has no convergent subsequence which contradicts to X being sequentially compact. □

Theorem 5.11. *Let (X, d) be a metric space. Then X is compact if and only if it is sequentially compact.*

Proof. The necessary condition has been obtained in Proposition 5.8. We are now going to the converse statement. Assume that X is sequentially compact. Let $\mathcal{U}$ be an open cover of X. Let λ be a Lebesgue number for the covering $\mathcal{U}$ given as in Lemma 5.9. Since X is sequentially compact, X is totally bound by Proposition 5.10, there are finitely many elements $x_1, ..., x_N$ in X such that $X = \bigcup_{k=1}^N B(x_k, \lambda)$. Since λ is a Lebesgue number for $\mathcal{U}$, for each $k = 1, ..., N$, there is $U_k \in \mathcal{U}$ such that $B(x_k, \lambda) \subseteq U_k$. Therefore we have $X = U_1 \cup \cdots \cup U_N$ and thus, X is compact. The proof is finished. □

Exercise 5.12. *If (X,d) is a compact metric space, then X is complete.*

The converse of Exercise 5.12 is clearly not true, for example when $X=\mathbb{R}$. In order to see under what condition of X so that the converse holds, let us introduce the following definition that is naturally led by Proposition 5.10.

Definition 5.13. *A metric space X is said to be* totally bounded *if for every $r>0$, there are finitely many elements $x_1,..x_N$ in X such that $X=\bigcup_{k=1}^N B(x_k,r)$.*

Theorem 5.14. *Let (X,d) be a metric space. Then X is compact if and only if X is complete and totally bounded.*

Proof. ($\Rightarrow$): Assume that X is compact and so X is sequentially compact. Thus, if (x_n) is a Cauchy sequence in X, then it has a convergent subsequence (x_{n_k}). Let $z:=\lim_k x_{n_k}$. Since (x_n) is Cauchy, we see that $\lim_n x_n=z$. Hence, X is complete. The totally boundedness can be clearly obtained by the compactness of X.
($\Leftarrow$): Assume that X is complete and totally bounded. Let (x_n) is a sequence in X. If the set $\{x_n : n=1,2...\}$ if finite, then clearly (x_n) has a convergent subsequence. Now we assume that the set $\{x_n\}$ is infinite. Since X is totally bounded, then there are finitely many open balls of radius $r_1:=1/2$, $B_1,..,B_N$ so that $X=B_1\cup\cdots B_N$. Then there is a B_i such that B_i contains an infinite subsequence $(x_k)_{k=1}^\infty$ of (x_n). By using the totally boundedness again, there is an open ball of radius $r_2:=\frac{1}{2^2}$ such that it contains an infinite subsequence, say (x_{2k}) of (x_{1k}). To repeat the same step inductively, for each $n=1,2..$, we have a sequence $(x_{nk})_{k=1}^\infty$ that satisfies the following properties:

(1) (i) $(x_{n+1,k})_{k=1}^\infty$ is a subsequence of (x_{nk}) for all $n=0,1,...$, where $x_{0k}:=x_k$.
(ii) $d(x_{nk},x_{nl})<\frac{2}{2^n}$ for all $k,l=1,2....$

Now put $y_n:=x_{nn}$ for $n=1,2....$ Then (y_n) is a subsequence of (x_n). Also by the condition (ii) above we see that (y_n) is a Cauchy sequence. This implies that (y_n) is convergent by the assumption of X being complete. Thus, X is sequentially compact, and so it is compact. □

In the rest of this section, we are going to study the product of compact spaces.

Definition 5.15. *Let X be a topological space.*

(i) A collection of closed subsets $\mathcal{F}$ is said to be a closed base if the collection $\{F^c : F\in\mathcal{F}\}$ forms an open base for X.
(ii) A collection of closed subsets $\mathcal{S}$ is said to be a closed subbase of X if the collection $\{S^c : S\in\mathcal{S}\}$ forms an open subbase for X.

Example 5.16. *Let $\{X_i\}$ be a collection of topological spaces. Let $X:=\prod_{i\in I}X_i$ be the product space endowed with the product topology. Recall that if we let $\pi_i : x\in X\mapsto x(i)\in X_i$ be the natural projection for each $i\in I$, then the product topology is the weakest topology so that each π_i is continuous.*
In this case, the collection

$$\mathcal{S}:=\{\pi_i^{-1}(F_i) : i\in I \text{ and } F_i \text{ is a closed subset of } X_i\}$$

forms a closed subbase of X.

Given a set E, a collection of subsets of E, say $\mathcal{F}$, is said to have *finite intersection property* if whenever for finitely many elements $F_1,..,F_n\in\mathcal{F}$ we have $F_1\cap\cdots\cap F_n\neq\emptyset$.
The following is directly shown by the DeMorgan's law.

Exercise 5.17. *Let X be a topological space. Then X is compact if and only if every collection of closed subsets of X with finite intersection property has non-empty intersection, that is, if whenever $\mathcal{F}$ is a collection of closed sets with finite intersection property, then $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.*

The following result is a useful characterization of compact spaces.

Proposition 5.18. *Let X be a topological space. Then the following are equivalent.*

(i) X is compact.

(ii) There is an open base $\mathcal{B}$ such that whenever a subfamily of $\mathcal{B}$ is an open cover of X has a finite subcover.

(iii) There is a closed base $\mathcal{C}$ such that whenever a subfamily of $\mathcal{C}$ with finite intersection property has nonempty intersection, that is, if $\mathcal{C} := \{C_i\}_{i\in I}$ is a closed base and a subfamily $\{C_j : j \in J\}$, where $J \subseteq I$ has finite intersection property, then $\bigcap_{j\in J} C_j \neq \emptyset$.

(iv) There is a closed subbase $\mathcal{S}$ such that whenever a subfamily of $\mathcal{S}$ with finite intersection property has nonempty intersection.

Proof. $(i) \Leftrightarrow (ii)$ follows directly from the definitions of compactness and open bases.
$(ii) \Leftrightarrow (iii)$ follows directly from the DeMorgan's law.
$(iii) \Rightarrow (iv)$ is clear because every closed base is also a closed subbase.
It suffices to show that $(iv) \Rightarrow (iii)$.
Let $\mathcal{S}$ be a closed subbase of X satisfying the condition (iv). Let $\mathcal{C}$ be the closed base generated by $\mathcal{S}$, that is every element $C \in \mathcal{C}$ has a form $S_1 \cup \cdots \cup S_n$ for some finitely many elements $S_1, ..., S_n \in \mathcal{S}$.
Let $\mathcal{C}_0$ be a subfamily of $\mathcal{C}$. Then by the Zorn's Lemma, we may assume that $\mathcal{C}_0$ is a maximal subcollection of $\mathcal{C}$ with finite intersection property, that is, if $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}$ has finite intersection property, then $\mathcal{C}_0 = \mathcal{C}_1$.
Claim 1: For each element $C \in \mathcal{C}_0$, if $C = S_1 \cup \cdots \cup S_N$ for some $S_1, ..., S_N \in \mathcal{S}$, then there exists S_k belonging to $\mathcal{C}_0$. To see this, assume that each $S_k \notin \mathcal{C}_0$. Then $\mathcal{C}_0 \subsetneq \mathcal{C}_0 \cup \{S_1\} \subseteq \mathcal{C}$. Then by the maximality, the collection $\mathcal{C}_0 \cup \{S_1\}$ does not have finite intersection property. Therefore, there are finite many elements $C_{11}, ..., C_{1q_1}$ in $\mathcal{C}_0$ such that $C_{11}\cap\cdots\cap C_{1q_1}\cap S_1 = \emptyset$. To repeat the same argument for each S_k, $k = 1, .., N$, then there is a finite sequence $C_{k1}, ..., C_{kq_k}$ in $\mathcal{C}_0$ such that $C_{k1}\cap\cdots\cap C_{kq_k} = \emptyset$.
This implies that

$$\bigcap_{k=1}^{N} \bigcap_{j=1}^{q_k} C_{kj} \cap C = \emptyset.$$

This leads to a contradiction because the collection $\mathcal{C}_0$ has finite intersection property. The **Claim1** follows.
Now if we put $\mathcal{C}_0 = \{C_i : i \in I\}$, then for each $i \in I$, **Claim 1** gives an element $S_i \in \mathcal{S}\cap\mathcal{C}_0$ and $S_i \subseteq C_i$.
Thus, the subcollection $\{S_i : i \in I\}$ of $\mathcal{S}$ has finite intersection property. Then by the assumption of (iv), we see that

$$\emptyset \neq \bigcap_{i\in I} S_i \subseteq \bigcap_{i\in I} C_i.$$

The proof is finished. ∎

We can now obtain one of important properties of compact spaces.

Theorem 5.19. Tychonoff's Theorem: *Let $\{X_i\}_{i\in I}$ be a collection of topological spaces. Then the product space $X := \prod_{i\in I} X_i$ is compact in product topology if and only if each X_i is compact.*

Proof. Let $\pi_i : X \to X_i$ be the natural projection defined as before, for $i \in I$. Then by the definition of the product topology, the necessary condition clearly follows from the fact that the continuous image

of a compact space is compact.
We are now going to show the converse. Assume that each X_i is compact. Then by the definition of product topology, the collection

$$\mathcal{S} := \{\pi_i^{-1}(F_i) : \ F_i \text{ is a closed subset of } X_i\}$$

forms a close subbase of X. By using Proposition 5.18, it suffices to show that whenever a subcollection $\mathcal{S}_0$ of $\mathcal{S}$ has finite intersection property, then $\mathcal{S}_0$ has non-empty intersection. To see this, write $\mathcal{S}_0 := \{S_\lambda : \lambda \in \Lambda\}$. Then for each $\lambda \in \Lambda$, there is $i_\lambda \in I$ such that $S_\lambda = \pi_{i_\lambda}^{-1}(F_\lambda)$ for a closed subset F_λ of X_{i_λ}. For each $i^* \in I$, put

$$\Lambda(i^*) := \{\lambda \in \Lambda : i_\lambda = i^*\}.$$

Notice that if $\Lambda(i^*) \neq \emptyset$, then the collection of closed subsets $\{F_\lambda : \lambda \in \Lambda(i^*)\}$ of X_{i^*} has finite intersection property. Then by the compactness of X_{i^*}, we see that $\bigcap_{\lambda\in\Lambda(i^*)} F_\lambda \neq \emptyset$. In this case, we fix any element $z(i^*) \in \bigcap_{\lambda\in\Lambda(i^*)} F_\lambda \subseteq X_{i^*}$. If $\Lambda(i^*) = \emptyset$, then we fix any element $x_{i^*} \in X_{i^*}$. Now for any element $i^* \in I$, we set

$$z^*(i^*) = \begin{cases} z(i^*) & \text{if } \ \Lambda(i^*) \neq \emptyset; \\ x_{i^*} & \text{otherwise .} \end{cases}$$

Then $z^* \in \bigcap_\lambda S_\lambda$. Hence, $\bigcap_{\lambda\in\Lambda} S_\lambda \neq \emptyset$. The proof is complete. □

6. $C(K)$ SPACES

Throughout this section, Let K be a compact Hausdorff space and let $C(K)$ be the space of all continuous functions ($\mathbb{R}$ or $\mathbb{C}$ valued) defined on K.

Exercise 6.1. *For each element $f \in C(K)$, put*

$$\|f\|_\infty := \{|f(x)| : x \in K\}.$$

Show that $C(K)$ is a complete normed space (a complete normed space is called a **Banach space***), that is, $\|\cdot\|_\infty$ is a norm function (see Definition 1.3) and $C(K)$ is complete under the metric given by the norm $\|\cdot\|_\infty$.*

Compact subsets of $C(K)$

From now on, we keep the notation as in Exercise 6.1. In this subsection, we further assume that (K,d) is a compact metric space. In the rest of this section, we are going to study the compact subspaces of $C(K)$.

Definition 6.2. *A subset F of $C(K)$ is said to be* equicontinuous *if for every $\varepsilon > 0$, there is $\delta > 0$ (depends on ε only) such that*

$$|f(x) - f(y)| < \varepsilon \quad \text{for all } f \in F \text{ and whenever } x, y \in K \text{ with } d(x,y) < \delta.$$

Exercise 6.3. *Show that every compact metric space is separable, that is it has a countable dense set.*

Theorem 6.4. Ascoli's Theorem*: Suppose that (K,d) is a compact metric space. Let F be a closed subset of $C(K)$. Then F is compact if and only if F is bounded, that is $\sup_{f\in F}\|f\|_\infty < \infty$, and is equicontinuous.*

Proof. ($\Rightarrow$): Assume that F is compact. From this clearly there are finitely many elements $f_1, .., f_N$ in F such that $F \subseteq B(f_1, 1) \cup \cdots \cup B(f_N, 1)$. Thus, for each element $f \in F$, we have $\|f - f_k\|_\infty < 1$ for some $k = 1, .., N$. This gives $\|f\|_\infty \le 1 + \|f_k\|_\infty \le 1 + L$ for all $f \in F$, where $L := \max\{\|f_i\|_\infty : i = 1, ..., N\} < \infty$. Therefore, F is bounded.
Next we are going to show that F is equicontinuous. Let $\varepsilon > 0$. Then by the compactness, there are finitely many elements $f_1, ..., f_N$ in F such that $F \subseteq B(f_1, \varepsilon) \cup \cdots \cup B(f_N, \varepsilon)$. On the other hand, since every continuous function defined on a compact set is uniformly continuous, so there is $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ whenever $x, x' \in K$ with $d(x, x') < \delta$ and for all $k = 1, ..., N$. Now for any $f \in F$, then $\|f - f_k\|_\infty$ for some $1 \le k \le N$. From this, if $d(x, x') < \delta$, then we have

$$|f(x) - f(x')| \le |f(x) - f_k(x)| + |f_k(x) - f_k(x')| + |f_k(x') - f(x')| < 3\varepsilon.$$

Therefore, F is equicontinuous as desired.
($\Leftarrow$): Assume that F is bounded and equicontinuous. Let (f_n) be a sequence in F. Since F is closed and $C(K)$ is complete, it suffices to show that (f_n) has a Cauchy subsequence.
Since K is a compact metric space, so it is separable by Exercise 6.3. Hence, K has a countable dense subset $\{e_1, e_2, ...\}$.
We first notice that since (f_n) is bounded, $(f_n(e_1))$ is a bounded sequence in $\mathbb{R}$. The Bolzano-Weierstrass Theorem gives a convergent subsequence (f_{1k}) of (f_n) such that $c_1 := \lim_k f_{1k}(e_1)$ exists. Next we consider the sequence $(f_{1k}(e_2))$. Then it is a bounded sequence in $\mathbb{R}$. By using the Bolzano-Weiestrass Theorem again, there is a subsequence (f_{2k}) of (f_{1k}) so that $c_2 := \lim_k f_{2k}(e_2)$ exists. Note

that we still have $\lim_k f_{2k}(e_1) = c_1$. To repeat the same step, for each $n = 1, 2..$, we have constructed a subsequence $(f_{nk})_{k=1}^{\infty})$ of (f_n) satisfying the following conditions:

(i) $(f_{n+1,k})_k$ is a subsequence of $(f_{nk})_k$ for all $n = 1, 2...$

(ii) $\lim_k f_{nk}(e_i)$ exists for all $i = 1, 2, ...$

Now put

$$g_n := f_{nn} \quad \text{for } n = 1, 2....$$

Then (g_n) is a subsequence of (f_n). We are going to show that (g_n) is a Cauchy sequence in $C(K)$. Let $\varepsilon > 0$. Then by the assumption of F being equicontinuous, there is $\delta > 0$ such that $|g_n(x) - g_n(y)| < \varepsilon$ for all $n = 1, 2, ...$ whenever $x, y \in K$ with $d(x, y) < \delta$. Since $\{e_i : i = 1, 2, ..\}$ is dense in K, we see that $K = \bigcup_i B(e_i, \delta)$. Then by the compactness, there is $i_0 \in \mathbb{N}$ such that $K = \bigcup_{i=1}^{i_0} B(e_i, \delta)$. Since $\lim_n g_n(e_i)$ exists for all $i = 1, 2..$, we can find n_0 such that $|g_n(e_i) - g_m(e_i)| < \varepsilon$ for all $m, n \geq n_0$ and for all $i = 1, 2..., i_0$. Now for any $m, n \geq n_0$, if $x \in B(e_i, \delta)$ for some $1 \leq i \leq i_0$, then we have

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(e_i)| + |g_n(e_i) - g_m(e_i)| + |g_m(e_i) - g_m(x)| < 3\varepsilon.$$

Hence, (g_n) is Cauchy. The proof is finished. □

In fact, in the proof of Theorem 6.4 above, we have shown the following.

Corollary 6.5. *Every bounded equicontinuous sequence in $C(K)$ has a convergent subsequence.*

Stone-Weierstrass Theorem

Notation: From now on, we will use the following notation.

Let K be a compact Hausdorff space as above and let $C(K, \mathbb{R})$ denote the space of all $\mathbb{R}$-valued continuous functions defined on K. Also, X is endowed with the $\|\cdot\|_\infty$-norm as before.

First let us recall the following classical Weierstrass approximation theorem first.

Theorem 6.6. Weierstrass Approximation Theorem: *Every real valued continuous function defined on $[a, b]$ can be uniformly approximated by a sequence of polynomial functions on $[a, b]$, that is for every $f \in C([a, b], \mathbb{R})$, there is a sequence of polynomial functions (p_n) on $[a, b]$ such that $\|f - p_n\|_\infty \to 0$.*

We are going to generalize this theorem to the case $C(K, \mathbb{R})$.

Now we are studying the order structure of $C(K, \mathbb{R})$ first. Recall that there is a natural order defined on $C(K, \mathbb{R})$, that is we say that $f \leq g$ for $f, g \in C(K, \mathbb{R})$ if $f(x) \leq g(x)$ for all $x \in K$. For $f, g \in C(K, \mathbb{R})$ and $x \in K$, put

$$(f \vee g)(x) := \max(f(x), g(x)) \quad \text{and} \quad (f \wedge g)(x) := \min(f(x), g(x)).$$

Notice that $f \vee g)$ and $f \wedge g$ both are continuous on K. In fact, it is clear that we have

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \quad \text{and} \quad f \wedge g = \frac{1}{2}(f + g - |f - g|). \tag{6.1}$$

Hence, $f \vee g = \sup\{f, g\}$ and $f \wedge g = \inf\{f, g\}$ under the natural order on $C(K, \mathbb{R})$.

A subset L of $C(K, \mathbb{R})$ is called a *lattice* whenever $f, g \in L$ implies that $f \vee g$ and $f \wedge g$ both are in L. In particular, $C(K, \mathbb{R})$ clearly is a lattice.

Proposition 6.7. *Let L be a vector subspace of in $C(K,\mathbb{R})$. Suppose that L satisfies the following conditions:*

(i) the constant function $1 \in L$, that is $1(x) = 1$ for all $x \in K$;
(ii) it is a lattice;
(iii) it separates the points in K, that is, for every pair of elements $x, y \in K$ with $x \neq y$, there is $f \in L$ so that $f(x) \neq f(y)$.

Then L is dense in $C(K,\mathbb{R})$, that is for every element $f \in C(K,\mathbb{R})$ and for all $\varepsilon > 0$, there is an element $\ell \in L$ such that $\|f - \ell\|_\infty < \varepsilon$.

Proof. We first claim that for any pair of elements $x, y \in K$ with $x \neq y$ and pair of real numbers a and b, then there is $g \in L$ such that $g(x) = a$ and $g(y) = b$.
To see this since $x \neq y \in K$, since L separates the points K, there is $h \in L$ such that $h(x) \neq h(y)$. Now if put $g := a\frac{h-h(y)}{h(x)-h(y)} - b\frac{h-h(x)}{h(x)-h(y)} = \frac{a-b}{h(x)-h(y)}h + \frac{bh(x)-ah(y)}{h(x)-h(y)}$, then $g \in L$ since L is a vector subspace and contains the constant function 1. In addition $g(x) = a$ and $g(y) = b$ as desired.
Now let $f \in C(K,\mathbb{R})$ and let $\varepsilon > 0$. We aim to find an element $\ell \in L$ such that $\|f - \ell\|_\infty < \varepsilon$.
We fix an element $x \in K$ first. Then by the above Claim, for every element $y \in K$, there is $g_y \in L$ such that

$$g_y(y) = f(y) \quad \text{and} \quad g_y(x) = f(x).$$

Put

$$G(y) := \{z \in K : g_y(z) < f(z) + \varepsilon\}.$$

Then $G(y)$ is an open neighbourhood of y. By the compactness of K, there are finitely many elements $y_1, .., y_N \in K$ such that $K = G(y_1) \cup \cdots \cup G(y_N)$. Now if we put

$$h_x := g_{y_1} \wedge \cdots \wedge g_{y_N}.$$

then $h_x \in L$ because L is lattice, in addition, we have

$$h_x(y) < f(y) + \varepsilon \quad \text{for all} \in X. \tag{6.2}$$

Now for each $x \in X$, set

$$V(x) := \{v \in X : f(v) - \varepsilon < h_x(v)\}.$$

Notice that since $g_y(x) = f(x)$ for all $y \in X$, we see that $h_x(x) = f(x)$. Thus, $V(x)$ is an open neighbourhood of x. Using the compactness again, there are finitely elements $x_1, ..., x_M$ such that $K = V(x_1) \cup \cdots \cup V(x_N)$. Now if we let

$$\ell := h_{x_1} \vee \cdots \vee h_{x_M}.$$

Then $\ell \in L$ and $\ell(x) > f(x) - \varepsilon$ for all $x \in K$. Also, by Eq 6.2 above, we have $\ell(x) < f(x) + \varepsilon$ for all $x \in K$. Hence we have $\|f - \ell\|_\infty < \varepsilon$ as desired. The proof is complete. □

Recall that a vector subspace S of $C(K,\mathbb{R})$ is called an *subalgebra* of $C(K,\mathbb{R})$ if whenever $f, g \in S$, then the usual product $f \cdot g \in S$, where $f \cdot g(x) := f(x)g(x)$ for $x \in K$.

Proposition 6.8. *If S is a closed subalgebra of $C(K,\mathbb{R})$, then it is a lattice.*

Proof. By using Eq 6.1, it suffices to show that if $f \in S$, then $|f| \in S$. Now let $0 \neq f \in S$. Clearly, we may assume that $\|f\|_\infty = 1$. Hence we have $-1 \leq f(x) \leq 1$ for all $x \in K$.
Let $g(t) := |t|$ for $t \in [-1,1]$. Then the Weierstrass approximation theorem gives a sequence of polynomials functions (p_n) on $[-1,1]$ such that $\|g - p_n\|_\infty \to 0$. In particular, we have $|g(0) - p_n(0)| = |p_n(0)| \to 0$. Therefore, we may assume that each $p_n(t)$ has no constant term. This implies that $p_n(f) \in S$ for all $n = 1, 2...$ since $-1 \leq f(x) \leq 1$ for all $x \in K$. Also, note that $g(f) = |f|$ on K. Thus, we have $||f(x)| - p_n(f)(x)| \to 0$ uniformly on K and hence, $\||f| - p_n(f)\|_\infty \to 0$. Since S is closed, $|f| \in S$. The proof is complete. □

We reach the following theorem of a generalization of Weierstrass Approximation Theorem.

Theorem 6.9. Stone-Weierstrass Theorem: *Let K be a compact Hausdorff space. Let A be a subalgebra of $C(K,\mathbb{R})$. If A separates the points in K and contains the constant function 1, then A is dense in $C(K,\mathbb{R})$.*

Proof. Note that $\overline{A}$ is a closed subalgebra, then by Proposition 6.8, $\overline{A}$ is a closed lattice. Proposition 6.7 implies that $\overline{A} = C(K,\mathbb{R})$. The proof is complete. □

Theorem 6.10. Complex Stone-Weierstrass Theorem: *Let K be a compact Hausdorff space. Let B be dense complex subalgebra of $C(K,\mathbb{C})$. Assume that B separates the points in K and contains the constant function 1. We further assume that $f \in B$ if and only its conjugate $\overline{f} \in B$. Then B is dense in $C(K,\mathbb{C})$.*

Proof. Let $A := \{h \in B : h(B) \subseteq \mathbb{R}\}$. Clearly, A is a real subalgebra of $C(K,\mathbb{R})$ and contains the constant function 1.
Now we want to show that A separates the points in K. Since B contains all conjugates of functions, we notice that $f \in B$ if and only if the real part $Re(f)$ and the imaginary part $Im(f)$ of f both are in A. From this, we see that A also separates the points K. Then one can apply the real case of Stone-Weierstrass Theorem, we see that B is dense in $C(K,\mathbb{C})$. The proof is complete. □

Urysohn's Lemma

In this section, we are going to show one of fundamental theorems about $C(K)$. The proof is divided by several steps.

Definition 6.11. *A topological space X is said to be normal if every disjoint closed subsets A and B of X, there are a pair of disjoint open neighbourhoods U and V such that $A \subseteq U$ and $B \subseteq V$.*

Exercise 6.12. *A topological space X is normal if and only if every open neighbourhood V of a closed subset F of X, there is an open neighbourhood W of F such that*

$$F \subseteq W \subseteq \overline{W} \subseteq V.$$

Proposition 6.13. *Every compact Hausdorff space is normal.*

Proof. To see this, let K be a compact Hausdorff space as above. Let A and B be the disjoint closed subsets of K. We first notice that A and B both are compact sets since K is compact Hausdorff. From this, we see that for each $x \in A$, there is a pair of disjoint open neighbourhoods of x and B respectively, say V_x and W_x (**Why?**). Then by the compactness of A, there are finitely many elements $x_1, ..., x_N$ in A such that $A \subseteq V := V_{x_1} \cup \cdots \cup V_{x_N}$. Thus, if we put $W := W_{x_1} \cap \cdots \cap W_{x_N}$, then V and W both are disjoint open neighbourhoods of A and B as desired. □

Lemma 6.14. *Let X be a topological space and let $\mathfrak{U} := \{U_i\}_{i \in I}$ be a family of open subsets of X, where I is a dense subset of $(0,1)$. Assume that the collection $\mathfrak{U}$ satisfies the following conditions:*

$$U_{i_1} \subseteq \overline{U_{i_1}} \subseteq U_{i_2} \quad \text{whenever } i_1 < i_2. \tag{6.3}$$

Now for each $x \in X$, put $I(x) := \{i \in I : x \in U_i\}$. If we define a map $f : X \to [0,1]$ by

$$f(x) := \begin{cases} \inf I(x) & \text{when } \quad I(x) \neq \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

then f is continuous.

Proof. Fix a point $x_0 \in X$. We are going to show that f is continuous at x_0. Let $c := f(x_0)$ and let $0 < \varepsilon < 1$.
We consider the case $I(x_0) \neq \emptyset$ and $c > 0$ first. Since I is a dense subset of $(0, 1)$, there are $i_1, i_2 \in I$ such that $c - \varepsilon < i_1 < c < i_2 < c + \varepsilon$. By using the Eq 6.3, one can directly check that we have $x_0 \in U_{i_2} \setminus \overline{U_{i_1}} \subseteq f^{-1}(c - \varepsilon, c + \varepsilon)$. Thus, f is continuous at x_0.
Next if $I(x_0) \neq \emptyset$ and $c = 0$, then $x_0 \in U_i$ for all $i \in I$. Now if we choose $i_0 \in I$ such that $0 < i_0 < \varepsilon$, then $x_0 \in U_{i_0} \subseteq f^{-1}(0 - \varepsilon, 0 + \varepsilon)$.
Finally, when $I(x_0) = \emptyset$, then $f(x_0) = 1$. Choose $i_1 \in I$ such that $1 - \varepsilon < i_1 < 1$. Using the Eq 6.3 again, we have $x_0 \in X \setminus \overline{U_{i_1}} \subseteq f^{-1}(1 - \varepsilon, 1 + \varepsilon)$ and so f is also continuous at x_0 in this case. The proof is complete. □

Theorem 6.15. Urysohn's Lemma: *If X is a normal space, then for every pair of disjoint non-empty closed subsets A and B, there is a continuous function $f : X \to [a, b]$ such that $f(A) \equiv a$ and $f(B) \equiv b$.*
In particular, the assertion holds for every compact Hausdorff space.

Proof. Notice that clearly $[0, 1]$ is homeomorphic to $[a, b]$, so we may assume that $[a, b] = [0, 1]$.
Let A and B be disjoint non-empty closed subsets of X. Then B^c is an open neighbourhood of A. Using exercise 6.12, there is an open subset $U_{1/2}$ of X such that

$$A \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq B^c.$$

Applying Exercise 6.12 for the pairs $A \subseteq U_{1/2}$ and $\overline{U_{1/2}} \subseteq B^c$ again, there are open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq B^c.$$

To repeat the same step, for each $n = 1, 2, ..,$ we have a finite family of open subsets of X, say $\{U_{k/2^n} : k = 1, 2..., 2^n - 1\}$ such that

$$A \subseteq U_{k/2^n} \subseteq \overline{U_{k/2^n}} \subseteq U_{(k+1)/2^n} \subseteq \overline{U_{(k+1)/2^n}} \subseteq B^c.$$

Now if we let $I := \bigcup_{n=1}^{\infty} \{k/2^n : k = 1, 2, .., 2^n - 1\}$. Then I is a dense subset of $(0, 1)$. Thus, we can apply Lemma 6.14 to yield a continuous map $f : X \to [0.1]$ such that $f(A) \equiv 0$ and $f(B) \equiv 1$ as required.
The last statement is obtained by Proposition 6.13 immediately. □

Remark 6.16. *Now if K is a compact Hausdorff space, then $C(K) \neq \mathbb{R}1$, that is $C(K)$ has non-constant functions. To see this, since K is Hausdorff, every one-point set is closed. The Urysohn's Lemma implies that whenever $a, b \in X$ with $a \neq b$, then there is $f \in C(K)$ such that $f(a) \neq f(b)$. Hence $C(K)$ has non-constant functions, in particular, it is "large" enough to separate the points in K.*

Theorem 6.17. Tietze extension theorem: *Let X be a normal space. If $f : M \to [a, b]$ is continuous function defined on a closed subset M of X, then there is a continuous extension of f on X, that is, there is a continuous function $F : X \to [a, b]$ such that $F(y) = f(y)$ for all $y \in M$.*

Proof. Clearly, we may assume that $[a, b] = [-1, 1]$. Put $f_0 := f$.
Let

$$A_0 : \{y \in M : f_0(y) \leq -1/3\} \quad \text{and} \quad B_0 := \{y \in M : 1/3 \leq f_0(y)\}.$$

Then by the Urysohn's Lemma, there is a continuous function

$$g_0 : X \to [-1/3, 1/3]$$

such that $g_0(A_0) = -1/3$ and $g_0(B_0) = 1/3$.
Now if we set $f_1(y) := f_0(y) - g_0(y)$, $y \in M$, then we have

$$f_1 : M \to [-2/3, 2/3] \quad \text{and} \quad |g_0(x)| \le 1/3, \quad x \in X.$$

Next we set

$$A_1 := \{y \in M : f_1(y) \le (-1/3)(2/3)\} \quad \text{and} \quad B_1 := \{y \in M : (1/3)(2/3) \le f_1(y)\}.$$

Using the Urysohn's again, there is a continuous function

$$g_1 : X \to [(-1/3)(2/3), (1/3)(2/3)]$$

such that $g_1(A_1) = (-1/3)(2/3)$ and $g_1(B_1) = (1/3)(2/3)$.
Now if we set $f_2(y) := f_1(y) - g_1(y)$, $y \in M$, then we have

$$f_2 : M \to [-(2/3)^2, (2/3)^2] \quad \text{and} \quad |g_1(x)| \le (1/3)(2/3), \quad x \in X.$$

Notice that we have

$$f_2(y) = f_1(y) - g_1(y) = f_0(y) - (g_0(y) + g_1(y)), \quad y \in M.$$

Similarly, we set

$$A_2 := \{y \in M : f_1(y) \le (-1/3)(2/3)^2\} \quad \text{and} \quad B_2 := \{y \in M : (1/3)(2/3)^2 \le f_1(y)\}.$$

Using the Urysohn's again, there is a continuous function

$$g_2 : X \to [(-1/3)(2/3)^2, (1/3)(2/3)^2]$$

such that $g_2(A_2) = (-1/3)(2/3)^2$ and $g_2(B_2) = (1/3)(2/3)^2$.
Now if we set $f_3(y) := f_2(y) - g_2(y)$, $y \in M$, then we have

$$f_3 : M \to [-(2/3)^3, (2/3)^3] \quad \text{and} \quad |g_2(x)| \le (1/3)(2/3)^2, \quad x \in X.$$

In addition, we have

$$f_3(y) = f_0(y) - (g_0(y) + g_1(y) + g_2(y)), \quad y \in M.$$

To do it inductively on n, we get the sequences of continuous functions

$$f_n : M \to [-(2/3)^n, (2/3)^n] \quad \text{and} \quad g_n : X \to [(1/3)(2/3)^n, (1/3)(2/3)^n]$$

$$f_{n+1} = f_0 - (\sum_{k=0}^{n} g_k) \quad \text{on } M. \tag{6.4}$$

Notice that since $|g_n| \le \sum_{k=0}^{n}(1/3)(2/3)^n$ on X for all n, then the series $\sum_{k=0}^{\infty} g_k$ converges uniformly on X by the M-test. Hence, if we let $F := \sum_{k=0}^{\infty} g_k$, then F is continuous on X.
On the other hand, the sequence (f_n) converges uniformly to 0 on M because $|f_(y)| \le (2/3)^n$ for all $y \in M$ and for all n. Thus, Eq6.4 gives $F(y) = f_0(y) := f(y)$ for all $y \in M$ and so F is a continuous extension of f as desired. □

7. Locally compact spaces

Definition 7.1. *A topological space X is said to be locally compact if every point in X has an open neighbourhood with compact closure.*

Example 7.2. *Clearly, every compact space is locally compact.*

(i) $\mathbb{R}^n$ *and* $\mathbb{C}^n$ *are locally compact.*

(ii) $\mathbb{R}^\infty$ *defined in Example 5.2 (iii) is not locally compact. To see this, notice that the translations, that is $x \mapsto x + a$ for some $a \in \mathbb{R}^\infty$, and dilations, that is $x \mapsto \alpha x$ for some $\alpha > 0$, are homeomorphisms from $\mathbb{R}^\infty$ to itself. Thus, if $\mathbb{R}^\infty$ is locally compact, then the closed unit ball $B := \{x \in \mathbb{R}^\infty : \rho(0, x) \leq 1\}$ is compact. It leads to a contradiction (see also Example 5.6).*

The following is one of important properties about locally compact Hausdorff spaces.

Proposition 7.3. *If X is a locally compact Hausdorff space, then for any x_0 in X and any open neighbourhood V of x_0, there is an open neighbourhood U of x_0 such that*

$$x_0 \in U \subseteq \overline{U} \subseteq V \quad \text{and} \quad \overline{U} \text{ is compact.} \tag{7.1}$$

Furthermore, if K is a compact subset of X and $\mathcal{O}$ is an open neighbourhood of K, then there is an open neighbourhood $\mathcal{V}$ of K with compact closure such that

$$K \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{O}.$$

Proof. We are going to show Eq 7.1 first. Let C be an neighbourhood of x_0 with compact closure $\overline{C}$. Then $\overline{C} \cap V$ is an open neighbourhood of x_0 in $\overline{C}$ with respect to the relative topology in $\overline{C}$. Since $\overline{C}$ is compact, Proposition 6.13 tells us that it is normal. Then by Exercise 6.12, there is an open neighbourhood W of x_0 (open in X) such that

$$x_0 \in W \cap \overline{C} \subseteq \overline{W \cap \overline{C}}^{\overline{C}} \subseteq C \cap V \subseteq V$$

where $\overline{(\cdot)}^{\overline{C}}$ denotes the closure in $\overline{C}$ with respect to the relative topology. Therefore, if we show that

$$\overline{W \cap C} \subseteq \overline{W \cap \overline{C}}^{\overline{C}}, \tag{7.2}$$

then we have

$$x_0 \in W \cap C \subseteq \overline{\overline{C} \cap V} \subseteq V.$$

Then the open neighbourhood $U := W \cap C$ of x_0 is as required in Eq 7.1. Thus, it suffices to show the Eq7.2 holds. To see this, if $z \in \overline{W \cap C}$, then for any open neighbourhood E of x_0 (open in X), we have

$$\emptyset \neq E \cap (W \cap C) \subseteq (E \cap \overline{C}) \cap (W \cap \overline{C}).$$

This implies that $z \in \overline{W \cap \overline{C}}^{\overline{C}}$.

For the last assertion, if K is a compact subset of X and $\mathcal{O}$ be an open neighbourhood of K, then by Eq 7.1 above, for each $x \in K$, there is an open neighbourhood U_x of x such that $x \in U_x \subseteq \overline{U_x} \subseteq \mathcal{O}$. Then by the compactness of K, there are finitely many elements $x_1, .., x_N$ in K such that $K \subseteq V_{x_1} \cup \cdots \cup V_{x_N}$. Notice that we always have $\overline{V_{x_1} \cup \cdots \cup V_{x_N}} \subseteq \overline{V_{x_1}} \cup \cdots \cup \overline{V_{x_N}}$ (**Why?**). Thus, if we put $\mathcal{V} := V_{x_1} \cup \cdots \cup V_{x_N}$, then $K \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{O}$.

The proof is finished. □

From now on, let X be a locally compact Hausdorff space.

Definition 7.4. *Let f be a continuous function ($\mathbb{R}$ or $\mathbb{C}$-valued) defined on X. The* support *of f is defined by the set*

$$supp(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

Write $C_c(X)$ for the set of all continuous functions defined on X with compact supports.

Remark 7.5. *Notice that every continuous function with compact support on X is bounded. Hence, one can define the* sup-*norm* $\|\cdot\|_\infty$ *on $C_c(X)$ by*

$$\|f\|_\infty := \sup_{x\in X} |f(x)|, \quad f \in C_c(X).$$

Then $(C_c(X), \|\cdot\|_\infty)$ is a normed space.

Exercise 7.6. *Is the space $(C_c(X), \|\cdot\|_\infty)$ complete for a general locally compact space X?*

Proposition 7.7. *Let K be a compact subset of a locally compact Hausdorff space X and $K \subseteq V$ be an open neighbourhood. Then there is $f \in C_c(X)$ with $0 \leq f \leq 1$ such that*

$$f(K) \equiv 1 \quad and \quad f(X \setminus V) \equiv 0.$$

Hence, $supp(f) \subseteq V$.

Proof. First, Proposition 7.3 implies that there is an open neighbourhood W of K with compact closure such that

$$K \subseteq W \subseteq \overline{W} \subseteq V.$$

Notice that K and $\overline{W} \setminus W$ is a pairwise disjoint closed subsets of $\overline{W}$. One can apply the Urysohn's Lemma on $\overline{W}$ because $\overline{W}$ is compact Hausdorff. Then there is an element $f_0 \in C(\overline{W}, [0,1])$ such that $f_0(K) \equiv 1$ and $f_0(\overline{W} \setminus W) \equiv 0$. If we define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} f_0(x) & \text{if} \quad x \in \overline{W} \\ 0 & \text{otherwise} \end{cases}$$

then $f \in C(X, [0,1])$ with $supp(f) \subseteq \overline{W}$ and thus, the function $f \in C_c(X)$ is as desired. ☐

The following result is always used in many areas of mathematics.

Proposition 7.8. *Let X be a locally compact space and K be a compact subset of X. For any finite open cover $\mathcal{U} := \{U_k\}_{k=1}^n$ of K, there exists a finite collection*

$$\{\varphi_k \in C_c(X, [0,1]) : supp(\varphi_k) \subseteq U_k;\ k = 1, 2..., n\}$$

such that

$$\sum_{k=1}^n \varphi_k = 1 \quad on\ K.$$

In this case the family $\{\varphi_k\}_{k=1}^n$ is called a partition of unity *for K.*

Proof. We first claim that there is a finite open cover $\{V_k\}_{k=1}^n$ of K such that each $\overline{V_k}$ is compact and

$$V_k \subseteq \overline{V_k} \subseteq U_k.$$

To see this, for each $x \in K$, then $x \in U_{k_x}$ for some $1 \leq k_x \leq n$. Using Proposition 7.3, one can find an open neighbourhood W_x of x such that $x \in W_x \subseteq \overline{W_x} \subseteq U_{k_x}$ and $\overline{W_x}$ is compact. Then by the compactness of K, there are finitely many $x_1, ..., x_m \in K$ such that $K \subseteq W_{x_1} \cup \cdots \cup W_{x_m}$. Now for each U_i, $i = 1, .., n$, let $V_i := \bigcup_{x_j \in U_i} W_{x_j}$. Then by the construction of W_{x_j}'s, we have

$$V_i \subseteq \overline{V_i} = \overline{\bigcup_{x_j \in U_i} W_{x_j}} \subseteq \bigcup_{x_j \in U_i} \overline{W_{x_j}} \subseteq U_i$$

for all $i = 1, .., n$. Thus, the family $\{V_i : i = 1, .., n\}$ is as desired. The claim follows.
Next, for each $i = 1, .., n$ we can apply Proposition 7.7 to find an element $f_i \in C_c(X, [0, 1])$ with $supp(f_i) \subseteq U_i$ such that $f_i(\overline{V_i}) \equiv 1$.
Now if we put $\varphi_1 := 1 - (1 - f_1)$, then $\varphi_1 = 1$ on V_1 and $supp(\varphi_1) \subseteq U_1$.
Next we let $\varphi_2 \in C(X)$ so that

$$\varphi_1 + \varphi_2 = 1 - (1 - f_1)(1 - f_2).$$

Then $\varphi_1 + \varphi_2 = 1$ on $V_1 \cup V_2$. Notice that if $x \notin \overline{W_2}$, then $\varphi_2(x) = 0$ and so $supp(\varphi_2) \subseteq U_2$. To repeat the same step, $\varphi_k \in C(X)$, $k = 1, 2..., n-1$ are found so that $\varphi_1 + \cdots + \varphi_{n-1} = 1$ on $V_1 \cup \cdots \cup V_{n-1}$ and $supp(\varphi_k) \subseteq U_k$. Now set $\varphi_n \in C(X)$ such that

$$\varphi_1 + \cdots \varphi_{n-1} + \varphi_n = 1 - (1 - f_1) \cdots (1 - f_{n-1})(1 - f_n) \tag{7.3}$$

Then $\varphi_1 + \cdots + \varphi_n = 1$ on $V_1 \cup \cdots \cup V_n$ and so, $\varphi_1 + \cdots + \varphi_n = 1$ on K. On the other hand, if $x \notin \overline{W_n}$, then Eq 7.3 implies that $\varphi_n(x) = 0$ and so $supp(\varphi_n) \subseteq U_n$. Thus, the family $\{\varphi_k : k = 1, .., n\}$ is the partition of unity as required. The proof is complete. □

The following is one of important features of locally compact Hausdorff spaces.
Recall that let $(X, \mathfrak{T})$ be a locally compact Hausdorsff space. Consider the following disjoint union:

$$X_\infty := X \bigsqcup \{\infty\}.$$

Let

$$\mathfrak{T}_\infty := \mathfrak{T} \quad \bigcup \quad \{X_\infty \setminus C : C \text{ is a compact subset of } X\}.$$

Exercise 7.9. *We keep the notation as above. Then $\mathfrak{T}_\infty$ is a compact Hausdorff topology on X_∞ and X is dense in X_∞.*
We call $(X_\infty, \mathfrak{T}_\infty)$ the one-point compactification of X.

Example 7.10. *$\mathbb{R}_\infty$ is homeomorphic to a circle. To see this, let $S := \{(x, y) : x^2 + (y - \frac{1}{2})^2 = (1/2)^2\}$. For each element $z \in S$ is parametrized by $z := (x, y) = (\cos\theta \sin\theta, \sin^2\theta)$, $\theta \in [0, \pi]$. Define*

$$f(z) = \begin{cases} \tan\theta & \textit{when } \theta \neq \pi/2; \\ \infty & \textit{otherwise}. \end{cases}$$

Then f is a homeomorphism from S onto $(\mathbb{R})_\infty$.

Let X be a locally compact space. Let $C_0(X)$ be the space of ($\mathbb{R}$ or $\mathbb{C}$ valued) continuous functions on X which vanish at infinity, that is, $f \in C_0(X)$ whenever for any $\varepsilon > 0$, there is a compact subset K of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$.

Proposition 7.11. *Define $T : f \in C_0(X) \mapsto \widetilde{f} \in C(X_\infty)$ by $\widetilde{f}(x) = f(x)$ when $x \in X$; and $\widetilde{f}(\infty) = 0$. Then T is an isometry, that is $\|Tf\|_\infty = \|f\|_\infty$ for $f \in C_0(X)$. Hence, $C_0(X)$ can be viewed as a closed subspace of $C(X_\infty)$ and so $C_0(X)$ is complete.*
Furthermore, $\overline{C_c(X)} = C_0(X)$. Consequently, $C_0(X)$ is the completion of $C_c(X)$.

Proof. One can directly show that T is an isometry (**Try!**). We are going to show that $C_c(X)$ is dense in $C_0(X)$. Let $f \in C_0(X)$. Let $\varepsilon > 0$. Then there is a compact subset K of X such that $|f(x)| < \varepsilon$ when $x \notin K$. Proposition 7.8 tells us that there is an element $f_1 \in C_c(X, [0,1])$ with such that $f_1(K) \equiv 1$. If we put $g(x) := f(x)f_1(x)$ for $x \in X$, then $g \in C_c(X)$. Notice that $|g(x) - f(x)| = 0$ if $x \in K$ and $|g(x) - f(x)| = |f(x)|(1 - f_1(x))| < \varepsilon$ when $x \notin K$. Therefore, we have $\|g - f\|_\infty < \varepsilon$. The proof is finished. □

8. Connected Spaces

Definition 8.1. *A topological space X is said to be connected if whenever a pair of disjoint open subsets A and B such that $X = A \cup B$, then either A or B is empty.*

Remark 8.2. Let X be a topological space.

(i) By the definition of connectedness, X is disconnected if and only if there is a pair of disjoint non-empty open subsets A and B of X such that $X = A \cup B$. Hence, X is disconnected if and only if there is a non-empty proper closed and open subset (also called **clopen set**) of X.

(ii) A subset Y of X is connected if it is connected in the relative topology. In this case, Y is a connected subset of X if and only if whenever a pair of open subsets A and B of X with $Y \subseteq A \cup B$ such that $(A \cap Y) \cap (B \cap Y) = \emptyset$, then $Y \subseteq A$ or $Y \subseteq B$.

Example 8.3. *We have the following assertions.*

(i) A subset of $\mathbb{R}$ is connected if and only if it is an interval. In particular, $\mathbb{Q}$ is disconnected. Alternatively, if we fix any irrational number c, then $\mathbb{Q} \cap [c, \infty) = \mathbb{Q} \cap (c, \infty)$ and so it is a proper non-empty clopen subset of $\mathbb{Q}$.

(ii) $\mathbb{R}^n$ and $\mathbb{C}^n$ are connected.

Proposition 8.4. *Let D be a subset of a topological space X. If D is connected, then so is its closure $\overline{D}$.*

Proof. Let A and B be a pair of open subsets of X such that $\overline{D} \subseteq A \cup B$ and $\overline{D} \cap A \cap B = \emptyset$. Since D is connected, we have $D \subseteq A$ or $D \subseteq B$. We may assume that $D \subseteq A$. Then $\overline{D} \subseteq A$. In fact, if there is a point $x_0 \in \overline{D} \setminus A$, then $x_0 \in B$ because $\overline{D} \subseteq A \cup B$. This implies that there is $z \in D \cap B$ since $z \in B$ is open. Thus, $z \in D \cap B \cap A$, so $\overline{D} \cap A \cap B \neq \emptyset$. It leads to a contradiction. □

Proposition 8.5. *Let $f : X \to Y$ be a continuous map between the topological spaces X and Y. If X is connected, then so is the image $f(X)$.*

Proof. It follows immediately from the fact that the pre-image of every open set is open under a continuous mapping. □

Definition 8.6. *A topological space X is said to be path connected if for every pair of points a and b is X, there is a continuous path on X from a to b, that is there is a continuous function $f : [0, 1] \to X$, then $f(0) = a$ and $f(1) = b$.*

Remark 8.7. *Clearly every path connected space is connected but the converse does not hold. For example, consider the set*

$$X := \{(x, \sin(1/x) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}.$$

Notice that since the subset $Y := \{(x, \sin(1/x) : 0 < x \leq 1\}$ is connected and $\overline{Y} = X$, so, the space X is connected by Proposition 8.4. This shows us that X is not path connected because there is no continuous path connecting with $(0, 0)$ and $(0, 1)$. However, the converse holds for any non-empty open connected subset of $\mathbb{R}^n$.

Proposition 8.8. *If A is a non-empty open connected subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, then A is path connected.*

Proof. Assume that A is not path connected. Then there is a pair of points a and b in A which can not be connected by any continuous path on A. Put

$$U := \{x \in A : \text{ there is a continuous path on } A \text{ from } a \text{ to } x\}.$$

Let $x_0 \in U$. Since A is open, there is $r > 0$ such that the open ball $B(x_0, r) \subseteq A$. On the other hand, clearly every open ball in $\mathbb{R}^n$ is path connected, so $B(x_0, r) \subseteq U$. Thus, U is a non-empty open set in $\mathbb{R}^n$. If we let $V := A \setminus U$, then $b \in V$ and so it is non-empty. Due to the same reason every open ball in $\mathbb{R}^n$ being path connected, V is an open subset of $\mathbb{R}^n$. Therefore, A is the union of a pair of disjoint non-empty open sets U and V which contradicts to the connectedness of A. The proof is complete. □

Definition 8.9. *A connected component C (or "component" for simply) of a topological space X means that it is a maximal connected subset of X, that is, C is connected and if A is a connected subset of X containing C, then $C = A$.*
Notice that every component is closed by Proposition 8.4 above.

Lemma 8.10. *Let $\{C_i\}_{i\in I}$ be a collection of connected subsets of X. If $\bigcap C_i \neq \emptyset$, then $\bigcup_i C_i$ is connected.*

Proof. Let A and B be a pair of open subsets of X such that $(\bigcup_i C_i) \subseteq A \cup B$ and $(\bigcup_i C_i) \cap A \cap B = \emptyset$. Notice that we have $C_i \cap A \cap B = \emptyset$ for all $i \in I$. Since C_i is connected, we have $C_i \subseteq A$ or $C_i \subseteq B$ for all $i \in I$. Now suppose that there are $p \neq q \in I$ such that $C_p \subseteq A$ and $C_q \subseteq B$. Let $z \in C_p \cap C_q$ since $\bigcap C_i \neq \emptyset$. This implies that $z \in C_p \cap A \cap B$ which leads to a contradiction because $C_i \cap A \cap B = \emptyset$ for all $i \in I$. Therefore, if $C_p \subseteq A$ for some $p \in I$, then $\bigcup_i C_i \subseteq A$. Thus, $\bigcup_i C_i$ is connected. □

Proposition 8.11. *Let X be a topological space. Then every point in X is contained in a unique component.*
Consequently, the collection of all components forms a partition of X, that is, if $\{C_i : i \in I\}$ is the collection of all components of X, then $X = \bigcup C_i$ and $C_i \cap C_j = \emptyset$ when $i \neq j$.

Proof. For each $z \in X$, let

$$\mathcal{C}_z := \{C \subseteq X : \ C \text{ is connected and } \ z \in C\}.$$

Notice that $\{z\} \in \mathcal{C}_z$ and so $\mathcal{C}_z \neq \emptyset$. Put $M_z := \bigcup_{C\in\mathcal{C}_z} C$. Then by Lemma 8.10 and the definition of $\mathcal{C}_z$, M_z is the unique component containing z as desired. The proof is complete. □

From now on, for each $x \in X$ we write $C(x)$ for the unique component containing x. In particular, $C(x)$ is the largest connected subset containing x.

Definition 8.12. *A topological X is said to be* **totally disconnected** *if $C(x) = \{x\}$ for all $x \in X$, that is every connected subset of X is a singleton.*

Remark 8.13. Clearly every discrete space, that is the space is endowed with the discrete topology, is totally disconnected, however, the converse is not true, for example, the set of all rational numbers $\mathbb{Q}$ is totally disconnected but it is non-discrete. To see this, let E be a non-empty subset of $\mathbb{Q}$ containing at least two points. Let $a, b \in E$ with $a < b$. Choose any irrational number c such that $a < c < b$. Then $E \subseteq (-\infty, c) \cup (c, \infty)$ and $E \cap (-\infty, c) \cap (c, \infty) = \emptyset$. Therefore, E is disconnected.
In fact, in this proof, we have obtained the following.

Proposition 8.14. *Let X be a topological space. If for every pair of distinct elements can be separated by disjoint clopen subsets of X, then X is totally disconnected.*

Example 8.15. *Let $X := \prod_{n=1}^{\infty} X_n$ and $X_n := \{0,1\}$. If X is endowed with product topology, then X is totally disconnected. In fact, let $\pi_n : X \to X_n$ be the natural projection. If $x \neq y \in X$, then $\pi_n(x) \neq \pi_n(y)$ for some n, Hence, x and y can be separated by the disjoint clopen subsets $\pi_n^{-1}(\{1\})$ and $\pi_n^{-1}(\{0\})$.*

From now on, X always denotes a Hausdorff space. Write $CO(X)$ for the set of all clopen subsets of X. For each element $x_0 \in X$, put

$$C(x_0) := \bigcap_{x_0 \in E \in CO(X)} E.$$

Notice that $C(x)$ is contained in any clopen neighbourhood of x because $C(x)$ is connected. Hence, we always have $C(x) \subseteq CO(x)$ for all $x \in X$.

Lemma 8.16. *We keep the notation as above. If X is compact Hausdorff, then $CO(x)$ is connected for all $x \in X$.*
Consequently, $CO(x) = C(x)$ for all x.

Proof. Fix an element $x \in X$. Suppose that

$$CO(x) = A_0 \cup B_0$$

for a pair of disjoint open subsets in $C(x)$ and $x \in A_0$. We need to show that $C(x) \subseteq A_0$. We first note that since $CO(x)$ is closed in X, we see that A_0 and B_0 are also closed in X. Recall a fact that every compact Hausdorff space in normal. Therefore, there is a pair of disjoint open sets in X, say A, B such that $A_0 \subseteq A$ and $B_0 \subseteq B$ and so we have

$$CO(x) \subseteq A \cup B.$$

This gives

$$\bigcap_{x \in E \in CO(X)} E \cap A^c \cap B^c = \emptyset.$$

Recall a fact that a topological space is compact if and only if it has finite intersection property. Therefore, there are finitely many clopen neighbourhoods of x , say $E_1, ..., E_n$ such that

$$\bigcap_{i=1}^{n} E_i \cap A^c \cap B^c = \emptyset.$$

Next we are going to show that $\bigcap_{i=1}^n E_i \cap A$ is a clopen neighbourhood of x. To see this, since each E_i is an clopen neighbourhood of x, so, $\bigcap_{i=1}^n E_i \cap A$ is open in X. It remains to show that $\bigcap_{i=1}^n E_i \cap A$ is closed in X. To see this, if there is a point

$$w \in \overline{\bigcap_{i=1}^{n} E_i \cap A} \setminus (\bigcap_{i=1}^{n} E_i \cap A),$$

then $w \in \overline{\bigcap_{i=1}^n E_i} = \bigcap_{i=1}^n E_i$ since E_i is closed. On the other hand, we have $\bigcap_{i=1}^n E_i \subseteq A \cup B$ because $\bigcap_{i=1}^n E_i \cap A^c \cap B^c = \emptyset$. Therefore, if $w \notin A$, then $w \in B$ and so $\bigcap_{i=1}^n E_i \cap B$ is an open neighbourhood of w. This gives $\bigcap_{i=1}^n E_i \cap B \cap A \neq \emptyset$ since $w \in \overline{A}$. It leads to a contradiction because $A \cap B = \emptyset$. Therefore, $\bigcap_{i=1}^n E_i \cap A$ is a clopen neighbourhood of x. Then by the definition of $CO(x)$, we have

$$CO(x) \subseteq \bigcap_{i=1}^n E_i \cap A \subseteq A.$$

This implies that if $y \in CO(x)$, then $y \notin B$ because $CO(x) \subseteq A \cup B$ and $A \cap B = \emptyset$. Therefore, $y \notin B_0$ since $B_0 \subseteq B$. Hence, $CO(x) \subseteq A_0$ as desired. The proof is complete. □

Theorem 8.17. *Let X be a compact Hausdorff space. We keep the notation as above. Then the following are equivalent.*

(i) X is totally disconnected, that is $C(x) = \{x\}$ for all $x \in X$.
(ii) $CO(x) = \{x\}$ for all $x \in X$.
(iii) For every pair of distinct elements $a, b \in X$, there are a pair of disjoint clopen neighbourhoods of a and b respectively.
(iv) The collection of all clopen sets $CO(X)$ forms an open base.

Proof. In fact, $(ii) \Rightarrow (i)$ holds for general topological spaces because we always have $C(x) \subseteq CO(x)$ for all $x \in X$.
$(i) \Rightarrow (ii)$ is obtained by Lemma 8.16 immediately.
$(iii) \Rightarrow (i)$ has been shown in Proposition 8.14 for general Hausdorff spaces.
For showing $(ii) \Rightarrow (iii)$, let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then by the assumption (ii), there are clopen sets E_1 and E_2 of x_1 and x_2 respectively so that $x_2 \notin x_1$ and $x_1 \notin E_2$. The $E_1 \setminus E_2$ and $E_2 \setminus E_1$ are the disjoint clopen neighbourhoods of x_1 and x_2 as required.
Thus, we have shown that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.
Now for $(iii) \Rightarrow (iv)$, let W be any open subset of X and thus, W^c is compact. We may assume that $W^c \neq \emptyset$. Let $x \in W^c$. Then for every element $y \in W$, the condition (iii) gives a pair of disjoint clopen neighbourhoods of x and y respectively. From this and the compactness of W^c, we can find a clopen neighbourhood U of x disjoint from W^c and so $U \subseteq W$.
$(iv) \Rightarrow (i)$ is clear since (iv) implies that if a subset contains more than two points, then it disconnected because X is Hausdorff.
The proof is finished. □

Appendix: Projective Limits

A partially order set $(I, \geq)$ is called a *directed set* if for every pair of elements $i_1, i_2 \in I$, one can find an element $i_3 \in I$ such that $i_3 \geq i_1$ and $i_3 \geq i_2$.

Definition 8.18. *Let I be a directed set and let $\{X_i\}_{i \in I}$ be a family of non-empty topological spaces. Let $\{\varphi_{ji} : X_j \to X_i : i, j \in I; i \leq j\}$ be a family of continuous functions which satisfies the conditions:*

(i) whenever $i, j, k \in I$ with $i \leq j \leq k$, then we have the decomposition

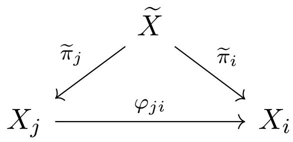
$$\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji} : X_k \to X_j \to X_i;$$

(ii) $\varphi_{ii} = Id_{X_i} : X_i \to X_i$.

In this case, the family $\{X_i, \varphi_{ij} : i, j \in I\}$ is called a *projective system (or inverse system).*
Now the product space $\prod_{i \in I} X_i$ is endowed with the product topology. Put

$$\varprojlim_{i \in I} X_i := \{(x_i) \in \prod_{i \in I} X_i : \varphi_{ji}(x_j) = x_i \quad \text{whenever } i \leq j\}.$$

For each $i \in I$, let $\widetilde{\pi}_i : \varprojlim_{i\in I} X_i \to X_i$ be the natural projection, that is $\widetilde{\pi}_i(x_j) = x_i$ for all i.
The set $\{\varprojlim_{i\in I} X_i, \widetilde{\pi}_i : i \in I\}$ is called the *projective limit (or inverse limit)* of the projective system $\{X_i, \varphi_{ji} : i, j \in I\}$. Notice that if $j \geq i$, then we always have the following commutative diagram:



where $\widetilde{X} := \varprojlim_{i\in I} X_i$.

The following is an important example of projective limits.

Example 8.19. *Let p be a prime number. For $m, n = 1, 2...$ with $n \leq m$, define a natural ring homomorphism*

$$\varphi_{mn} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$$

given by

$$\varphi_{m,n} : x + p^m\mathbb{Z} \mapsto x + p^n\mathbb{Z}.$$

Then the collection $\{\mathbb{Z}/p^n\mathbb{Z}; \varphi_{mn} : m, n = 1, 2...\}$ forms a projective system. Write $\mathbb{Z}_p$ for its projective limit. Call $\mathbb{Z}_p$ the **ring of p-adic integers** *which plays an important role in number theory.*

Definition 8.20. *A topological space X is said to be profinite if it is the projective limit of finite sets, that is, there is a projective system $\{X_i, \varphi_{ji} : i, j \in I\}$, where each X_i is a finite set such that X is homeomorphism to the projective limit of the system $\varprojlim_{i\in I} X_i$.*

Theorem 8.21. *X is a totally disconnected compact Hausdorff space if and only if it is profinite. In particular, $\mathbb{Z}_p$ is a totally disconnected compact Hausdorff space.*

Proof. We show the converse first. Assume that X is the projective limit of a projective system $\{X_i, \varphi_{ji} : i, j \in I\}$, where each X_i is a finite set. By the construction of a projective limit, we may assume that $X \subseteq \prod_{i\in I} X_i$. We keep the notation as in Definition 8.18.
Since each X_i is finite and so it is compact, the product space $\prod_{i\in I} X_i$ is a compact Hausdorff space. In order to show that X is compact, it suffices to show that the projective limit is closed in $\prod_{i\in I} X_i$. To see this, let $w \notin \varprojlim_{i\in I} X_i$. If we write $w = (w_i)$, then $\varphi_{ji}(w_j) \neq w_i$ for some i, j with $i \leq j$. Then $\pi_i^{-1}(w_i) \cap \pi_j^{-1}(w_j)$ is an open neighbourhood of w and $\pi_i^{-1}(w_i) \cap \pi_j^{-1}(w_j) \cap X = \emptyset$ and so X is closed in $\prod X_i$.
On the other hand, as in the proof of Example 8.15, each pair of distinct elements in X can be separated by a pair of disjoint clopen sets. Therefore, X is totally disconnected.
Now we are going to show the necessary condition. Assume that X is a totally disconnected compact Hausdorff space. We say that an equivalence relation R on X is clopen if each equivalence class of R is a clopen subsets of X. We write X/R for the set of all equivalent classes of R. Notice that since X is compact, X/R is a finite set when R is an clopen relation on X. Let

$$\mathcal{R} := \{R_i : \ R_i \text{ is a clopen equivalence relation on } X \ \}_{i\in I}.$$

For each $i \in I$, let $\pi_i : X \to X/R_i$ be the canonical projection, that is $\pi_i(x) = xR_i \in X/R_i$ for $x \in X$. For $i_1, i_2 \in I$, we say that $i_1 \leq i_2$ if $xR_{i_2} \subseteq xR_{i_1}$ for all $x \in X$. Then $(I, \geq)$ becomes a directed set. Moreover, for $i_1 \leq i_2$ in I, there is a natural function $\varphi_{i_2,i_1} : X/R_{i_2} \to X/R_{i_1}$ given by $xR_{i_2} \mapsto xR_{i_1}$. On can directly check that $\{X/R_i, \varphi_{i,j} : i, j \in I\}$ is a projective system. Let $\widetilde{X}$ be its projective limit constructed as in Definition 8.18 above. Define a map by

$$\psi : x \in X \mapsto (\pi_i(x))_{i \in I} \in \widetilde{X}.$$

We are going to the map ψ is a homeomorphism. Since X is compact and $\widetilde{X}$ is Hausdorff, it suffices to show that ψ is a continuous bijection.

We first claim that ψ is well defined, that is $\psi(x) \in \widetilde{X}$ for all $x \in X$. In fact, we have $\varphi_{i_2,i_1}(\pi_{i_2}(x)) = \pi_{i_1}(x)$ for all $x \in X$ whenever $i_1 \leq i_2$ in I. Thus, $\psi(x) \in \widetilde{X}$.

For showing the map ψ is injective, let $x \neq y \in X$. Since X is totally disconnected compact Hausdorff, Theorem 8.17 gives a clopen set W such that $x \in W$ but $y \notin W$. If we let $R := \{W, W^c\}$, then $R \in \mathcal{R}$ and $\pi_R(x) \neq \pi_R(y)$. Thus, $\psi(x) \neq \psi(y)$.

Before showing the map ψ being surjective, we have to show that ψ is continuous. To see this, recall the construction of the projective limit in Definition 8.18 that $\widetilde{\pi}_i : \widetilde{X} \to X/R_i$ is the canonical projection for each $i \in I$. Consider the composition $\widetilde{\pi}_i \circ \psi : X \to \widetilde{X} \to X/R_i$ for each $i \in I$. Since each X/R_i is discrete and the $(\widetilde{\pi}_i \circ \psi)^{-1}(xR_i) = \{y \in X : yR_i x\}$ is a clopen subset of X for all $xR_i \in X/R_i$, so the map ψ is continuous.

This implies that the image $\psi(X)$ is compact and thus, $\psi(X)$ is closed in $\widetilde{X}$ because $\widetilde{X}$ is Hausdorff. Therefore, if we can show that the image $\psi(X)$ is dense in $\widetilde{X}$, then ψ is surjective.

In order to show that $\psi(X)$ is dense in $\widetilde{X}$. Let $\widetilde{x} = (\widetilde{x}_i) \in X$, $\widetilde{x}_i \in X/R_i$, for $i \in I$. Then by the definition of the product topology, it suffices to show that for any finitely many $i_1, ..., i_n \in I$, we want to find an element $z \in X$ such that

$$\psi(z)_{i_k} := zR_{i_k} = \widetilde{x}_{i_k} \quad \text{for all} \quad k = 1, ..., n.$$

Since I is a directed set, there is $j \in I$ such that $j \geq i_k$ for all $k = 1, ..., n$. Then $\widetilde{x}_j \in X/R_j$ satisfies the condition: $\varphi_{ji_k}(\widetilde{x}_j) = \widetilde{x}_{i_k}$ for all $k = 1, ..., n$.

Now take $z \in X$ such that $zR_j = \widetilde{x}_j \in X/R_j$. Then by the definition of ψ, we have

$$\psi(z)_{i_k} := zR_{i_k} = \varphi_{ji_k}(zR_j) = \varphi_{ji_k}(\widetilde{x}_j) = \widetilde{x}_{j_k} \quad \text{for all} \quad k = 1, .., n.$$

The proof is complete. □

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