EPYMT TDG 2025 Group 1 Tutorial

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Example 1 (Cycloid)**.**

$$\gamma(t) = (t - \sin t, 1 - \cos t)$$

for $t \in (0, 2\pi)$. To find its curvature κ, we notice that

$$\gamma'(t) = (1 - \cos t, \sin t)$$

and

$$|\gamma'|^2 = 2 - 2\cos t.$$

Hence, the unit vector is

$$T = \frac{1}{\sqrt{2}}(\sqrt{1-\cos t}, \frac{\sin t}{\sqrt{1 - \cos t}}) = (\sin \frac{t}{2}, \cos \frac{t}{2}).$$

It follows that

$$\frac{1}{2} = \left| \frac{\mathrm{d}T}{\mathrm{d}t} \right| = \left| \frac{\mathrm{d}s}{\mathrm{d}t} \right| \kappa = |\gamma'|\kappa.$$

Therefore,

$$\kappa = \frac{1}{2|\gamma'|} = \frac{1}{2^{\frac{3}{2}}\sqrt{1 - \cos t}}.$$

Example 2 (Helix)**.**

$$\gamma(t) = (\cos t, \sin t, t)$$

for $t \in \mathbb{R}$. To find its curvature κ, we notice that

$$\gamma'(t) = (-\sin t, \cos t, 1)$$

and $|\gamma'| = \frac{1}{\sqrt{2}}$. Hence,

$$T = \frac{\gamma'}{|\gamma'|} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1).$$

Finally,

$$\kappa = \frac{1}{\sqrt{2}|\gamma'|} = \frac{1}{2}.$$

Example 3 (Catenary)**.**

$$\gamma(t) = (t, \cosh t)$$

for $t \in \mathbb{R}$. To find its curvature κ, notice that $|\gamma'| = \cosh t$ and

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \left(-\frac{\sinh t}{\cosh^2 t}, \overset{2}{\operatorname{sech}} t \right).$$

Hence,

$$\kappa = \frac{1}{|\gamma'|}\left| \frac{\mathrm{d}T}{\mathrm{d}t} \right| = \frac{1}{\cosh^2 t}.$$

Example 4 (Tractrix)**.**

$$\gamma(t) = (\operatorname{sech} t, t - \tanh t)$$

for $t > 0$. Alternatively, we can also parametrize by

$$\gamma(t) = \left(\sin t, \ln\left(\cot\frac{t}{2}\right) - \cos t\right)$$

for $t \in (0, \frac{\pi}{2})$.

For the first parametrization, we have

$$\gamma' = \tanh t(-\operatorname{sech} t, \tanh t)$$

and

$$|\gamma'| = \tanh t.$$

Hence,

$$T = (-\operatorname{sech} t, \tanh t)$$

and

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \operatorname{sech} t(\tanh t, \operatorname{sech} t).$$

Finally,

$$\kappa = \frac{1}{\cosh t}\frac{\cosh t}{\sinh t} = \frac{1}{\sinh t}.$$

Alternatively, the second parametrization gives

$$\gamma'(t) = \left(\cos t, -\frac{\cos^2 t}{\sin t}\right)$$

and

$$|\gamma'| = \left|\frac{\cos t}{\sin t}\right|.$$

These combined gives $T = (\sin t, -\cos t)$ and therefiore

$$1 = \left|\frac{\mathrm{d}T}{\mathrm{d}t}\right| = \kappa\left|\frac{\cos t}{\sin t}\right|.$$

So,

$$\kappa = |\tan t|.$$

Theorem 1. *Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then, its curvature is*

$$\kappa = \frac{||\gamma'' \times \gamma'||}{||\gamma'||^3}.$$

Proof. Let γ' be derivative with respect to t and $s(t)$ be the arc-length element. By chain rule,

$$\begin{aligned}\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\gamma}{\mathrm{d}s} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}\gamma}{\mathrm{d}t} + \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{|\gamma'|}\cdot\frac{\mathrm{d}\gamma}{\mathrm{d}t} + \frac{1}{|\gamma'|^2}\frac{d^2\gamma}{dt^2}.\end{aligned}$$

For the derivative of $|\gamma'|^{-1}$, notice that it is well-defined because the curve is regular.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{|\gamma'|} &= \frac{1}{|\gamma'|^2}\left(-\frac{\mathrm{d}}{\mathrm{d}s}|\gamma'|\right) \\ 2|\gamma'|\frac{\mathrm{d}}{\mathrm{d}s}|\gamma'| = \frac{\mathrm{d}}{\mathrm{d}s}|\gamma'|^2 &= 2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\cdot\frac{\mathrm{d}\gamma}{\mathrm{d}t} \\ \therefore \frac{\mathrm{d}}{\mathrm{d}s}|\gamma'| &= \frac{1}{|\gamma'|}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\cdot\frac{\mathrm{d}\gamma}{\mathrm{d}t} \\ &= \frac{1}{|\gamma'|}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t} \\ &= \frac{1}{|\gamma'|^2}\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}.\end{aligned}$$

Combine all,

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{|\gamma'|} = -\frac{1}{|\gamma'|^4}\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t},$$

and hence,

$$\begin{aligned}\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2} &= -\frac{1}{|\gamma'|^4}\langle\gamma'',\gamma'\rangle\gamma' + \frac{1}{|\gamma'|^2}\gamma'' \\ &= \frac{1}{|\gamma'|^4}\left(-\langle\gamma'',\gamma'\rangle\gamma' + \langle\gamma',\gamma\rangle\gamma''\right).\end{aligned}$$

Recall that by brute force calculation there is the identity

$$a\times b\times c = \langle a,c\rangle b - \langle a,b\rangle c$$

for vector triple products. Hence,

$$\gamma'\times\gamma''\times\gamma' = \langle\gamma',\gamma'\rangle\gamma'' - \langle\gamma',\gamma'\rangle\gamma'.$$

Now, we arrive at

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2} = \frac{\gamma'\times\gamma''\times\gamma'}{|\gamma'|^4}.$$

Since in general we have $|a||b||\sin\theta| = |a\times b|$, we deduce $|\gamma'\times\times\gamma'| = |\gamma'||\gamma''\times\gamma'|$ because $\gamma'\perp\gamma''\times\gamma'$. Simplifying the fraction, by the definition of κ we can see that

$$\kappa = \frac{|\gamma''\times\gamma'|}{|\gamma'|^3}.$$

□

Remarks. For more exercises, read Shifrin's book. It is available online.

Exercise 1. Find the Frenet frame for

$$\gamma(t) = \frac{1}{\sqrt{3}}(e^t, e^t \cos t, e^t \sin t),$$

for $t \in \mathbb{R}$.

Exercise 2. Chapter 2 exercises in the lecture notes.