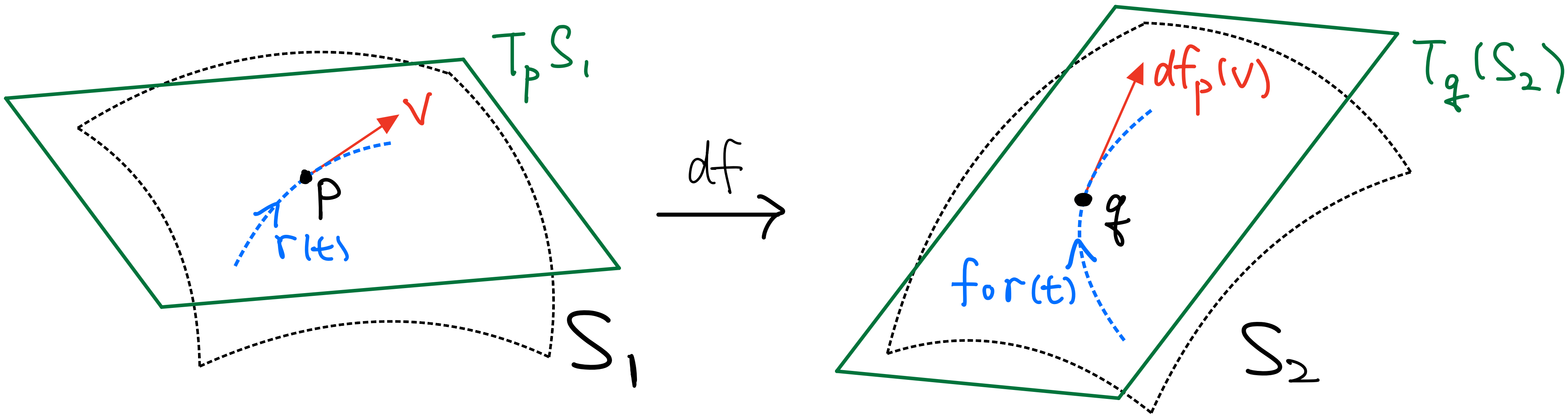
Definition (Differential) Suppose $f: S_1 \to S_2$. Let $p \in S_1$, $q = f(p) \in S_2$

Define $df_p : T_pS_1 \to T_qS_2$ as follow:

For $v \in T_pS_1$, let $\vec{r}(t)$ be a curve on S_1 with $\vec{r}(0) = p$, $\vec{r}\,'(0) = v$

Define $df_p(v) = (f \circ r)'(0) \in T_qS_2$



Theorem $df_p : T_pS_1 \to T_qS_2$ is linear: For any $\alpha, \beta \in \mathbb{R}$

$$df_p(\alpha X_u + \beta X_v) = \alpha\, df_p(X_u) + \beta\, df_p(X_v)$$

Definition 3.4.5 (Differential of Gauss map). *Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u,v)$. For each $p \in S$, define* $d\mathbf{n}_p : T_pS \to T_pS$ *called the* **differential of Gauss map** *by*

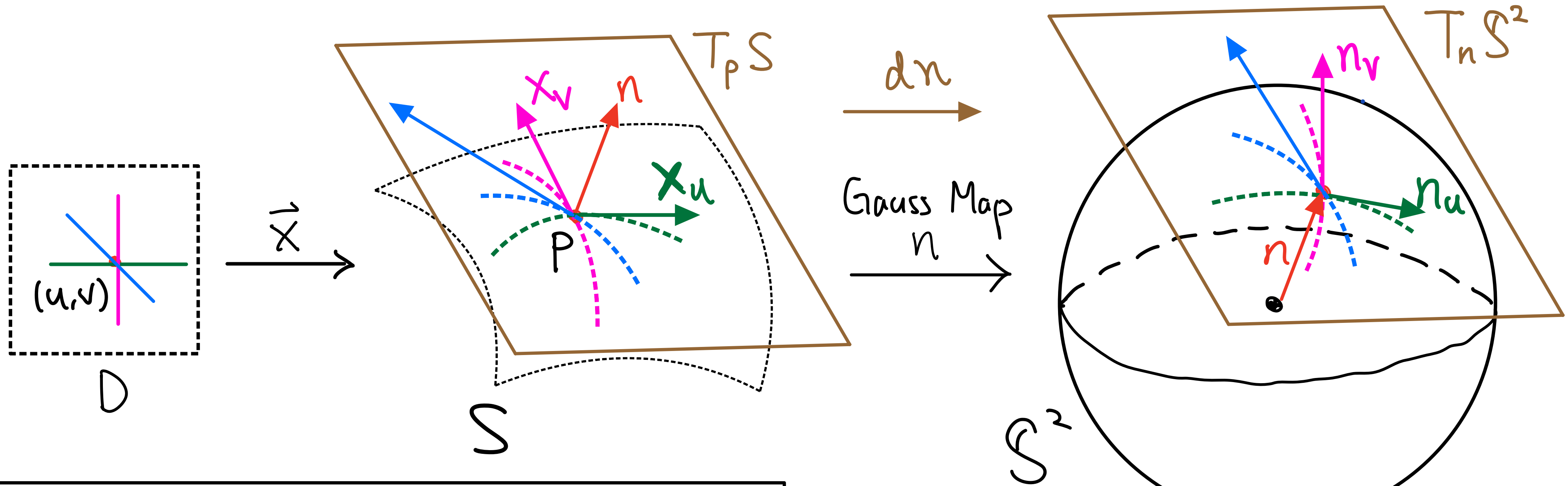
$$d\mathbf{n}_p(\alpha\mathbf{x}_u + \beta\mathbf{x}_v) = \alpha\mathbf{n}_u + \beta\mathbf{n}_v$$

for any real numbers $\alpha, \beta \in \mathbb{R}$.

Rmk

$dn_p(x_u) = n_u$

$dn_p(x_v) = n_v$



Proposition 3.4.2.

- $T_\mathbf{n}S^2 = T_pS$.
- $\mathbf{n}_u, \mathbf{n}_v \in T_pS$ $\Rightarrow \begin{cases} n_u = a x_u + b x_v \\ n_v = c x_u + d x_v \end{cases}$

Next :

$\boxed{\text{Express } a, b, c, d \text{ in terms of } \mathrm{I}, \mathrm{II}}$

Express a, b, c, d in terms of I, II

$$\begin{cases} n_u = a x_u + b x_v & \cdots \text{①} \\ n_v = c x_u + d x_v & \cdots \text{②} \end{cases}$$

$$\text{①} \Rightarrow \langle x_u, n_u \rangle = \langle x_u, a x_u + b x_v \rangle$$
$$= a \langle x_u, x_u \rangle + b \langle x_u, x_v \rangle$$
$$= aE + bF$$

$$\langle x_v, n_u \rangle = \langle x_v, a x_u + b x_v \rangle$$
$$= aF + bG$$

Similarly

$$\text{②} \Rightarrow \langle x_u, n_v \rangle = cE + dF$$
$$\langle x_v, n_v \rangle = cF + dG$$

$$-\mathrm{II} = \begin{bmatrix} \langle x_u, n_u \rangle & \langle x_v, n_u \rangle \\ \langle x_u, n_v \rangle & \langle x_v, n_v \rangle \end{bmatrix}$$

$$= \begin{bmatrix} aE + bF & aF + bG \\ cE + dF & cF + dG \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathrm{I}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -(\mathrm{II})(\mathrm{I}^{-1})$$

$$= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$= -\frac{1}{EG - F^2} \begin{pmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{pmatrix}$$

Rmk

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^2 \frac{\det \mathrm{II}}{\det \mathrm{I}} = K$$

Proposition 3.4.9. *The matrix representation of $d\mathbf{n}_p$ with respect to basis $\mathbf{x}_u, \mathbf{x}_v$ is*

$$-(II)(I^{-1}) = -\frac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF \\ fG-gF & gE-fF \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It means

$$d\mathbf{n}_p(\mathbf{x}_u) = a\mathbf{x}_u + b\mathbf{x}_v \qquad \text{where} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -(II)I^{-1}$$
$$d\mathbf{n}_p(\mathbf{x}_v) = c\mathbf{x}_u + d\mathbf{x}_v$$

Theorem 3.4.7 (Self-adjointness of differential of Gauss map)**.** *The differential of Gauss map $d\mathbf{n}_p : T_pS \to T_pS$ is self-adjoint. In other words, for any $\mathbf{u}, \mathbf{v} \in T_pS$, we have*

$$\langle d\mathbf{n}_p(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, d\mathbf{n}_p(\mathbf{v})\rangle.$$

Pf Let $u = \alpha X_u + \beta X_v \qquad v = \gamma X_u + \delta X_v$

$$\langle dn_p(u), v\rangle = \langle \alpha\, dn_p(X_u) + \beta\, dn_p(X_v), \gamma X_u + \delta X_v\rangle$$
$$= \alpha\gamma \langle dn_p(X_u), X_u\rangle + \alpha\delta \langle dn_p(X_u), X_v\rangle$$
$$+ \beta\gamma \langle dn_p(X_v), X_u\rangle + \beta\delta \langle dn_p(X_v), X_v\rangle$$

$$\langle u, dn_p(v)\rangle = \alpha\gamma \langle X_u, dn_p(X_u)\rangle + \alpha\delta \langle X_u, dn_p(X_v)\rangle$$
$$+ \beta\gamma \langle X_v, dn_p(X_u)\rangle + \beta\delta \langle X_v, dn_p(X_v)\rangle$$

Only need to check if $\langle dn_p(X_u), X_v\rangle = \langle X_u, dn_p(X_v)\rangle$:

$$\langle dn_p(X_u), X_v\rangle = \langle n_u, X_v\rangle \qquad \leftarrow \quad \langle n, X_v\rangle \equiv 0$$
$$= \langle n, X_{vu}\rangle \qquad \Rightarrow \frac{\partial}{\partial u}\langle n, X_v\rangle \equiv 0$$
$$= \langle n, X_{uv}\rangle \qquad \Rightarrow \langle n_u, X_v\rangle + \langle n, X_{vu}\rangle \equiv 0$$
$$= \langle n_v, X_u\rangle$$
$$= \langle X_u, n_v\rangle$$
$$= \langle X_u, dn_p(X_v)\rangle$$

1.5 Eigenvalues, eigenvectors and diagonalization

Definition 1.5.1 (Eigenvalues and eigenvectors)**.** *Let A be an $n \times n$ matrix. If λ is a complex number*[3] *and $\boldsymbol{\xi}$ is a non-zero*[4] *complex vector such that*

$$A\boldsymbol{\xi} = \lambda\boldsymbol{\xi},$$

then we say that λ is an **eigenvalue** *of A and $\boldsymbol{\xi}$ is an* **eigenvector** *of A associated with λ.*

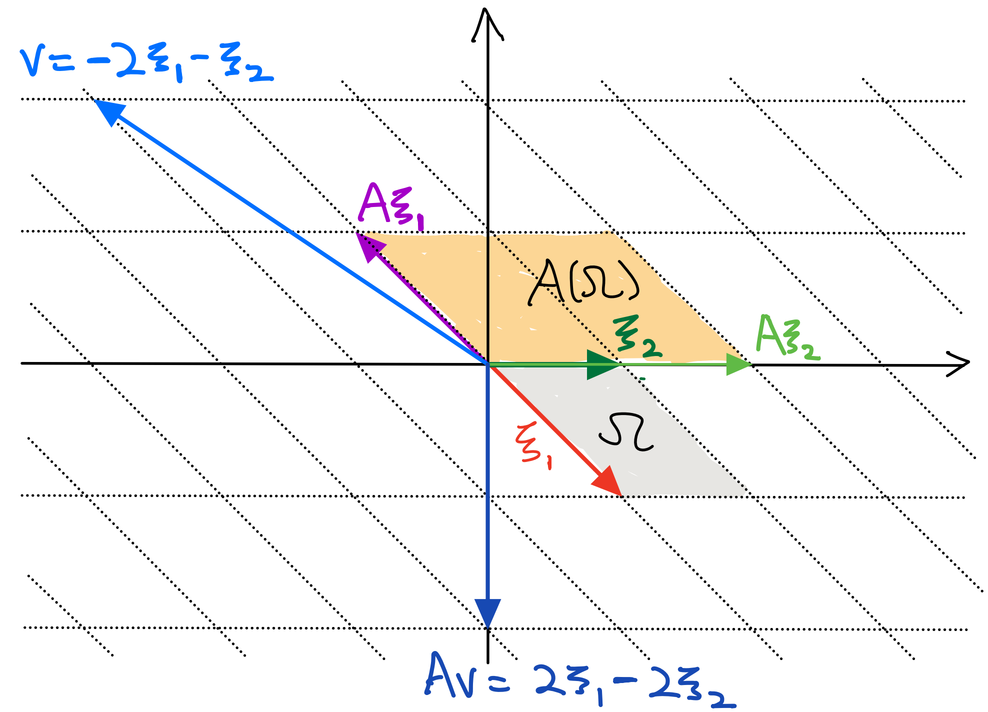
eg $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\xi_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \lambda_1 = -1$

$\xi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \lambda_2 = 2$



Eigenvectors help to understand Av

Proposition 1.5.3. *Let A be an $n \times n$ matrix.*

1. *A complex number λ is an eigenvalue of A if and only if λ is a root to the characteristic equation $\det(xI - A) = 0$.*
2. *Let λ be an eigenvalue of A. Then $\boldsymbol{\xi}$ is an eigenvector of A associated with λ if and only if $\boldsymbol{\xi} \neq \mathbf{0}$ and $(\lambda I - A)\boldsymbol{\xi} = \mathbf{0}$.*

eg $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

$$\det(xI - A) = \begin{vmatrix} x-1 & -2 \\ 1 & x-4 \end{vmatrix}$$

$$= (x-1)(x-4) - (-2)(1)$$

$$= x^2 - 5x + 6$$

$$= (x-2)(x-3)$$

$\det(xI - A) = 0$

$\Rightarrow x = 2$ or 3

Eigenvalues: 2, 3

For $\lambda_1 = 2$, $2I - A = \begin{bmatrix} 2-1 & -2 \\ 1 & 2-4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x - 2y = 0 \quad$ Take $\xi_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For $\lambda_2 = 3$, $3I - A = \begin{bmatrix} 3-1 & -2 \\ 1 & 3-4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x - y = 0 \quad$ Take $\xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector associated with eigenvalue 2

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector associated with eigenvalue 3

We can define eigenvalue, eigenvector for $dn_p : T_pS \to T_{n(p)}(S^2) = T_pS$ similarly

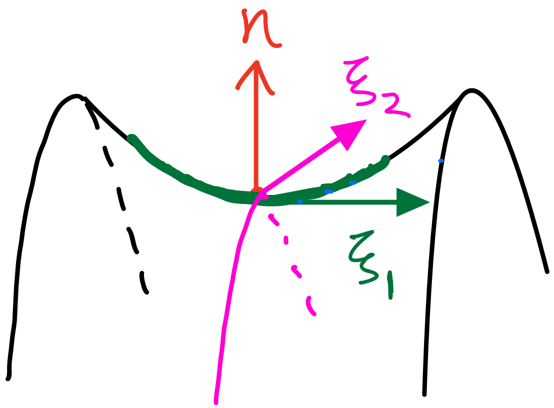
Theorem 3.4.8. *Let S be a regular surface in $\mathbb{R}^3$ and $p \in S$. Then there exists principal directions $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T_pS$ which constitute an orthonormal basis for T_pS.*

$$\begin{cases} d\mathbf{n}_p(\boldsymbol{\xi}_1) = -\kappa_1 \boldsymbol{\xi}_1 \\ d\mathbf{n}_p(\boldsymbol{\xi}_2) = -\kappa_2 \boldsymbol{\xi}_2 \end{cases}$$

Then we say that κ_1, κ_2 are the **principal curvatures** *of S at p, and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are the corresponding* **principal directions**.

<u>Rmk</u> dn_p is self-adjoint $\Longrightarrow$ Orthonormal basis exists (Linear Algebra: Section 1.4, 1.5)

<u>eg</u>



$$dn_p(\xi_1) = \overbrace{-\kappa_1}^{<0} \xi_1 \qquad \kappa_1 > 0$$

$$dn_p(\xi_2) = \underbrace{-\kappa_2}_{>0} \xi_2 \qquad \kappa_2 < 0$$

$\kappa > 0$ if the curve bends towards $\vec{n}$

Matrix representation of dn_p

Note $\{\xi_1, \xi_2\}$, $\{X_u, X_v\}$ are both basis of T_pS

$$dn_p(\xi_1) = -k_1\xi_1 + 0\,\xi_2 \qquad dn_p(X_u) = aX_u + bX_v$$

$$dn_p(\xi_2) = 0\,\xi_1 + (-k_2)\xi_2 \qquad dn_p(X_v) = cX_u + dX_v$$

Matrix representation of dn_p with respect to these basis

$$\begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-\mathrm{II})(\mathrm{I})^{-1} = -\frac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF \\ fG-gF & gE-fF \end{pmatrix}$$

Fact: The two matrices have same determinant and trace

$$k_1k_2 = \begin{vmatrix} -k_1 & 0 \\ 0 & -k_2 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^2\frac{\det \mathrm{II}}{\det \mathrm{I}} = K$$

$$-(k_1+k_2) = a+d = -\frac{gE-2fF+eG}{EG-F^2}$$

Theorem 3.4.10. *Let S be a regular surface and K be the Gaussian curvature of S. Then for any $p \in S$,*

$$K(p) = \det(d\mathbf{n}_p) = \kappa_1\kappa_2$$

Definition 3.4.11 (Mean curvature)**.** *Let S be a regular surface and $d\mathbf{n}_p$ be the differential of Gauss map at $p \in S$. The* **mean curvature** *of S at p is*

$$H = -\frac{1}{2}\mathrm{tr}(d\mathbf{n}_p) = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\mathrm{tr}((II)(I^{-1})) = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right).$$

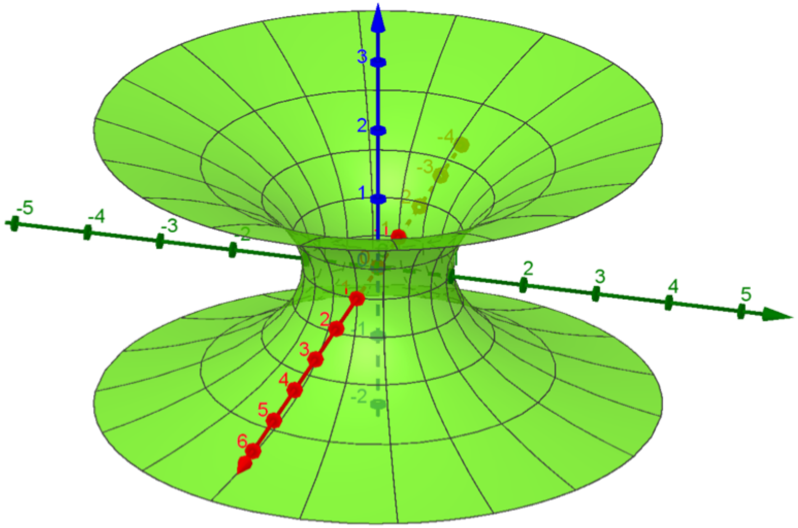
Definition 3.4.12 (Minimal surface)**.** *Let S be a regular surface in $\mathbb{R}^3$ and H be the mean curvature of S. We say that S is a* **minimal surface** *if $H = 0$ at every point of S.*

Example 3.4.14. *Show that the catenoid parametrized by*

$$\mathbf{x}(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v), \quad \underset{0}{\cancel{1}} < \theta < 2\pi, v \in \mathbb{R},$$

is a minimal surface.

$$\begin{aligned}
\mathbf{x}_\theta &= (-\cosh v \sin\theta, \cosh v \cos\theta, 0) \\
\mathbf{x}_v &= (\sinh v \cos\theta, \sinh v \sin\theta, 1) \\
\mathbf{x}_\theta \times \mathbf{x}_v &= (\cosh v \cos\theta, \cosh v \sin\theta, -\cosh v \sinh v) \\
\|\mathbf{x}_\theta \times \mathbf{x}_v\|^2 &= \cosh^2 v + \cosh^2 v \sinh^2 v = \cosh^2 v(1 + \sinh^2 v) = \cosh^4 v \\
\mathbf{n} &= (\operatorname{sech} v \cos\theta, \operatorname{sech} v \sin\theta, \tanh v) \\
\mathbf{x}_{\theta\theta} &= (-\cosh v \cos\theta, -\cosh v \sin\theta, 0) \\
\mathbf{x}_{\theta v} &= (-\sinh v \sin\theta, \sinh v \cos\theta, 0) \\
\mathbf{x}_{vv} &= (\cosh v \cos\theta, \cosh v \sin\theta, 0).
\end{aligned}$$



Then the first and second fundamental forms are

$$\begin{aligned}
I &= \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix} \\
II &= \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Thus the mean curvature is

$$H = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) = \frac{1}{2}\left(\frac{\cosh^2 v - \cosh^2 v}{\cosh^4 v}\right) = 0.$$

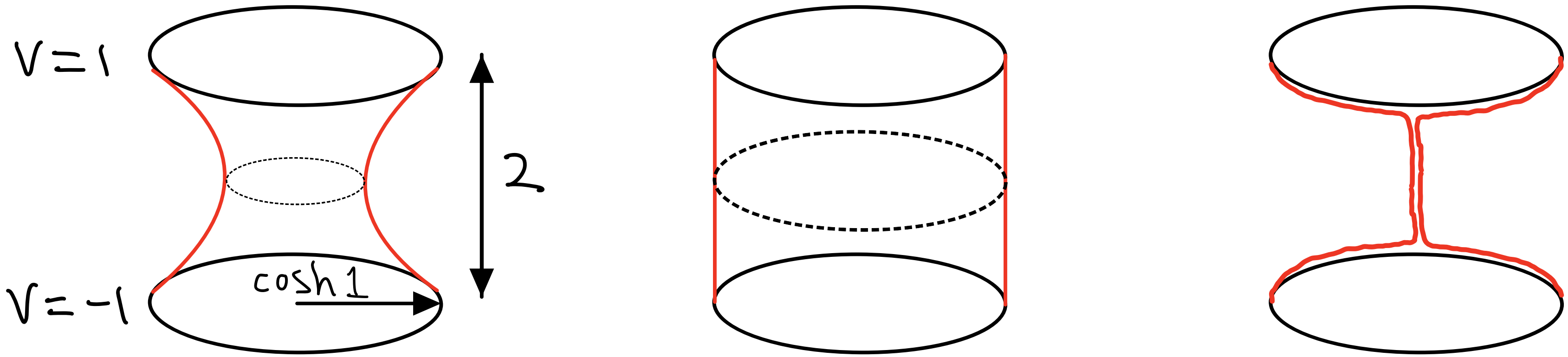
Therefore the catenoid is a minimal surface.

> **Theorem 3.4.13.** *Let S be a minimal surface with parametrization $\mathbf{x} : D \to \mathbb{R}^3$ such that $\mathbf{x}$ can be extended continuously to the boundary. Then S has the minimum surface area among all surfaces with the same boundary of S.*

eg Consider the part of the catenoid

$$\mathbf{x}(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v), \quad -1 < v < 1, \quad 0 < \theta < 2\pi$$

Catenoid　　　　Cylinder



	Catenoid	Cylinder	
Surface area	$2\pi(\cosh^2 1 - 1)$	$4\pi \cosh 1$	$\approx 2\pi \cosh^2 1$
	≈ 8.68	≈ 19.39	≈ 14.96

Theorem 3.4.19. *Let S be a regular surface parametrized by $\mathbf{x}(u,v)$ and K be the Gaussian curvature of S.*

1. $$K = \frac{\det(II)}{\det(I)}$$

   where I and II are the first fundamental forms of S.

2. $$\mathbf{n}_u \times \mathbf{n}_v = K\mathbf{x}_u \times \mathbf{x}_v$$

   where $\mathbf{n}$ is the unit normal vector of S.

3. $$K = \frac{d\sigma}{dA}$$

   where A and σ are the signed area function on S and S^2 respectively.

4. $$K = \kappa_1\kappa_2$$

   where κ_1, κ_2 are the principal curvatures associated with two orthogonal principal directions.

The following theorems relate curvature of a curve on S with κ_1, κ_2, II

Theorem 3.4.15. *Let S be a regular surface and $p \in S$ be a point on S. Let C be a regular parametrized curve passing through p. Then we have*

$$\kappa \cos\phi = -\langle \mathbf{T}, d\mathbf{n}_p(\mathbf{T}) \rangle$$

where $\mathbf{T}, \kappa$ are the unit tangent vector, signed curvature of C at p respectively, $d\mathbf{n}_p$ is the differential of Gauss map of S at p and ϕ is the angle between the unit normal vector $\mathbf{N}$ of C and the unit normal vector $\mathbf{n}$ of S at p. Furthermore if $\mathbf{T} = \alpha \mathbf{x}_u + \beta \mathbf{x}_v \in T_pS$, then we have

$$\kappa \cos\phi = (\alpha \quad \beta)\, II \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where II is the second fundamental form.

Definition 3.4.16 (Normal curvature). *Let S be a regular surface and p be a point on S. Let $\mathbf{v} \in T_pS$ be a unit vector tangent to the surface S at p. The* **normal curvature** *of S at p along $\mathbf{v}$ is*

$$\kappa_n(\mathbf{v}) = \kappa \cos\phi = -\langle \mathbf{v}, d\mathbf{n}_p(\mathbf{v}) \rangle$$

where κ is the curvature of a curve C which passes through p and has unit tangent vector equals to $\mathbf{v}$, and ϕ is the angle between the unit normal vectors $\mathbf{N}$ and $\mathbf{n}$ of C and S at p respectively.

Theorem 3.4.17.

$$\kappa_n(\mathbf{v}) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

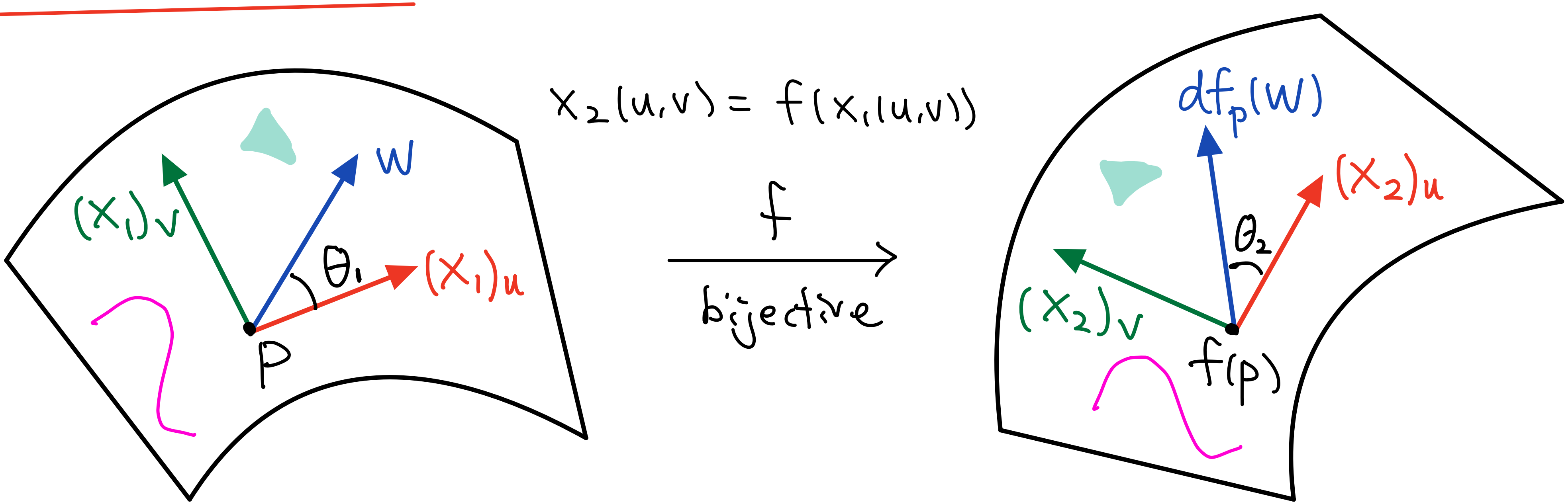
Theorem 3.4.18.

$$\kappa_1 \leq \kappa_n(\mathbf{v}) \leq \kappa_2.$$

3.5 Theorema egregium

Let S_1 be a regular surface and $f : S_1 \to S_2$ be a differentiable bijective map from S_1 to another regular surface S_2. Then any regular parametrization $\mathbf{x}_1(u,v)$ of S_1 induces a parametrization of S_2 by $\mathbf{x}_2(u,v) = f \circ \mathbf{x}_1(u,v) = f(\mathbf{x}_1(u,v))$. Furthermore the first fundamental forms $I_1(u,v)$ and $I_2(u,v)$ on S_1 and S_2 with respect to $\mathbf{x}_1(u,v)$ and $\mathbf{x}_2(u,v)$ can both be considered as matrix valued functions of u, v. We say that $f : S_1 \to S_2$ is an isometry if $I_1(u,v) = I_2(u,v)$ for any u, v.

$$I = \begin{pmatrix} \langle x_u, x_u \rangle & \langle x_u, x_v \rangle \\ \langle x_v, x_u \rangle & \langle x_v, x_v \rangle \end{pmatrix}$$



$$I_1 = I_2 \implies \begin{aligned} \|(x_1)_u\| &= \|(x_2)_u\| \\ \|(x_1)_v\| &= \|(x_2)_v\| \\ \|w\| &= \|df_p(w)\| \end{aligned}$$

$\theta_1 = \theta_2$

preserve arclength, area ..

Theorem 3.5.2 (Theorema egregium). *Let S_1 and S_2 be two regular surfaces. Suppose S_1 and S_2 are isometric, that is, there exists isometry $f : S_1 \to S_2$ between S_1 and S_2. Then for any $p \in S_1$, the Gaussian curvature of S_1 at p is equal to the Gaussian curvature of S_2 at $f(p)$. In other words,*

$$K(f(p)) = K(p)$$

for any $p \in S_1$.

Pf It follows from Theorem 3.5.4:

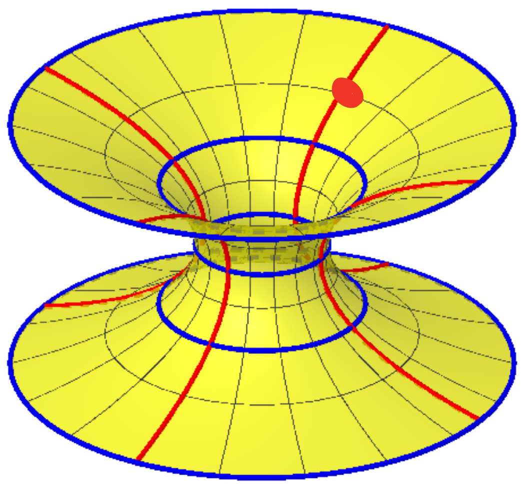
$$K = \frac{1}{4(EG - F^2)^2}\left(\begin{vmatrix} -E_{vv} + 2F_{uv} - G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & E & F \\ G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{vmatrix} \right)$$

Rmk We defined K using $\vec{n}$, I, II

Thm 3.5.2 $\Rightarrow$ K is actually independent of $\vec{n}$, II

Example 3.5.3 (Isometry between catenoid and helicoid)

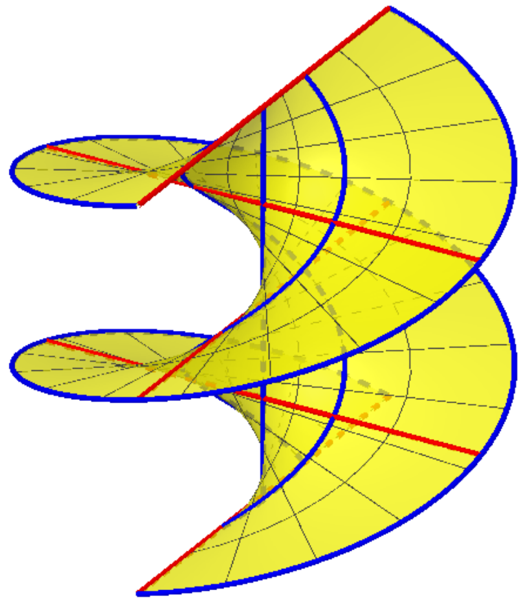
Catenoid



$$\xrightarrow{\quad f \quad}$$

$$f(x_1(\theta,v)) = x_2(\theta,v)$$

Helicoid

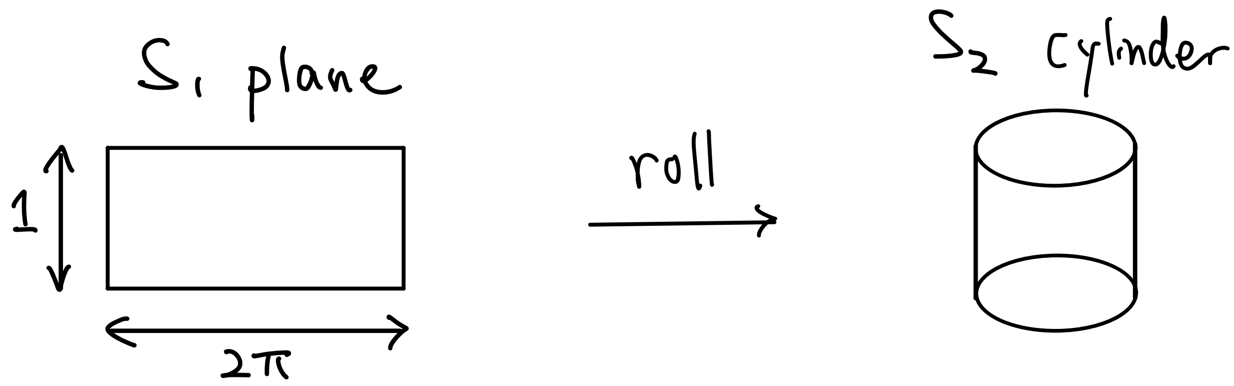


$\mathbf{x}_1(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v)$ $\quad (\theta, v) \in (0, 2\pi) \times \mathbb{R}$ $\quad \mathbf{x}_2(\theta, v) = (\sinh v \cos\theta, \sinh v \sin\theta, \theta)$

$$I_1(\theta,v) = I_2(\theta,v) = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}$$

$$\Rightarrow f \text{ is an isometry} \Rightarrow K_1(\theta,v) = K_2(\theta,v) = -\frac{1}{\cosh^4 v}$$

Catenoid and Helicoid are both minimal surface and thus have mean curvature identically zero. However, the mean curvature of two isometric surfaces may not be identical. For example, a cylindrical surface and a plane are isometric but a cylindrical surface has nonzero mean curvature while that of a plane is zero.



$X_1(\theta, z) = (\theta, 0, z)$ $\quad X_2(\theta, z) = (\cos\theta, \sin\theta, z)$ $\quad (\theta, z) \in (0, 2\pi) \times (0, 1)$

$$I_1 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow f \text{ is isometry} \Longrightarrow \text{same } K(\theta, z)$$

<u>Rmk</u> Different mean curvature $\quad H_1 = 0 \quad H_2 = \frac{1}{2}$ (or $-\frac{1}{2}$)

Theorem 3.5.4. *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface. Then*

$$K = \frac{1}{4(EG-F^2)^2}\left(\begin{vmatrix} -E_{vv}+2F_{uv}-G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & E & F \\ G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{vmatrix}\right).$$

In particular, if $F = 0$ *is identically zero, then*

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Proof. Since $\det(I) = EG - F^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2$ and $\|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n} = \mathbf{x}_u \times \mathbf{x}_v$,

$$\begin{aligned}
& K(EG-F^2)^2 \\
= \; & \det(I)\det(II) \\
= \; & \|\mathbf{x}_u \times \mathbf{x}_v\|^2 \begin{vmatrix} \langle \mathbf{x}_{uu}, \mathbf{n}\rangle & \langle \mathbf{x}_{uv}, \mathbf{n}\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n}\rangle & \langle \mathbf{x}_{vv}, \mathbf{n}\rangle \end{vmatrix} \\
= \; & \begin{vmatrix} \langle \mathbf{x}_{uu}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle & \langle \mathbf{x}_{uv}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle \\ \langle \mathbf{x}_{vu}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle & \langle \mathbf{x}_{vv}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle \end{vmatrix} \\
= \; & \begin{vmatrix} \langle \mathbf{x}_{uu}, \mathbf{x}_u \times \mathbf{x}_v\rangle & \langle \mathbf{x}_{uv}, \mathbf{x}_u \times \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_u \times \mathbf{x}_v\rangle & \langle \mathbf{x}_{vv}, \mathbf{x}_u \times \mathbf{x}_v\rangle \end{vmatrix} \\
= \; & \begin{vmatrix} \langle \mathbf{x}_{uu}, \mathbf{x}_{vv}\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle & \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{vmatrix} \\
& - \begin{vmatrix} 0 & \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_v\rangle & \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{vmatrix}. \text{ (Proposition 1.3.17)}
\end{aligned}$$

Observe that by product rule (Proposition 1.3.34),

$$\begin{aligned}
\begin{pmatrix} E_u & F_u \\ F_u & G_u \end{pmatrix} &= \frac{\partial}{\partial u}\begin{pmatrix} E & F \\ F & G \end{pmatrix} \\
&= \frac{\partial}{\partial u}\begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{vu}, \mathbf{x}_v\rangle \end{pmatrix} + \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_{uu}\rangle & \langle \mathbf{x}_u, \mathbf{x}_{vu}\rangle \\ \langle \mathbf{x}_v, \mathbf{x}_{uu}\rangle & \langle \mathbf{x}_v, \mathbf{x}_{vu}\rangle \end{pmatrix} \\
&= \begin{pmatrix} 2\langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle \\ \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & 2\langle \mathbf{x}_{vu}, \mathbf{x}_v\rangle \end{pmatrix}.
\end{aligned}$$

Similarly

$$\begin{pmatrix} E_v & F_v \\ F_v & G_v \end{pmatrix} = \begin{pmatrix} 2\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle & 2\langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle \end{pmatrix}.$$

Combining the above two equalities, we obtain

$$\begin{cases}
\langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle = \dfrac{E_u}{2}, \\
\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = \dfrac{E_v}{2}, \\
\langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = \dfrac{G_u}{2}, \\
\langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle = \dfrac{G_v}{2}, \\
\langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle = F_u - \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = F_u - \dfrac{E_v}{2}, \\
\langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle = F_v - \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = F_v - \dfrac{G_u}{2}.
\end{cases}$$

Moreover by considering the second derivative of $F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle$ with respect to u, v, we have

$$\begin{aligned}
\langle \mathbf{x}_{uu}, \mathbf{x}_{vv}\rangle &= \frac{\partial}{\partial u}\langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle - \langle \mathbf{x}_{vvu}, \mathbf{x}_u\rangle \\
&= \frac{\partial}{\partial u}\left(F_v - \frac{G_u}{2}\right) - \left(\frac{\partial}{\partial v}\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle\right) \\
&= F_{uv} - \frac{G_{uu}}{2} - \left(\frac{\partial}{\partial v}\frac{E_v}{2} - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle\right) \\
&= -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} + \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle
\end{aligned}$$

which implies

$$\langle \mathbf{x}_{uu}, \mathbf{x}_{vv}\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle = -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2}.$$

Therefore

$$K(EG-F^2)^2 = \left(\begin{vmatrix} -\dfrac{E_{vv}}{2} + F_{uv} - \dfrac{G_{uu}}{2} & \dfrac{E_u}{2} & F_u - \dfrac{E_v}{2} \\ F_v - \dfrac{G_u}{2} & E & F \\ \dfrac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \dfrac{E_v}{2} & \dfrac{G_u}{2} \\ \dfrac{E_v}{2} & E & F \\ \dfrac{G_u}{2} & F & G \end{vmatrix} \right)$$

as desire. If particular, if $F = 0$, then

$$\begin{aligned}
K &= \frac{1}{4E^2G^2}\left(\begin{vmatrix} -E_{vv} - G_{uu} & E_u & -E_v \\ -G_u & E & 0 \\ G_v & 0 & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & 0 \\ G_u & 0 & G \end{vmatrix} \right) \\
&= \frac{1}{4E^2G^2}\left(-EGE_{vv} - EGG_{uu} + GE_uG_u + EE_vG_v + GE_v^2 + EG_u^2\right) \\
&= -\frac{E_{vv}}{4EG} - \frac{G_{uu}}{4EG} + \frac{E_uG_u}{4E^2G} + \frac{E_vG_v}{4EG^2} + \frac{E_v^2}{4E^2G} + \frac{G_u^2}{4EG^2}.
\end{aligned}$$

Observe that

$$\begin{cases}
\left(\dfrac{E_v}{\sqrt{EG}}\right)_v = \dfrac{E_{vv}}{\sqrt{EG}} - \dfrac{E_v^2}{2E\sqrt{EG}} - \dfrac{E_vG_v}{2G\sqrt{EG}} \\
\left(\dfrac{G_u}{\sqrt{EG}}\right)_u = \dfrac{G_{uu}}{\sqrt{EG}} - \dfrac{G_u^2}{2G\sqrt{EG}} - \dfrac{E_uG_u}{2E\sqrt{EG}}.
\end{cases}$$

Hence

$$\begin{aligned}
&\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u \\
&= \frac{E_{vv}}{\sqrt{EG}} - \frac{E_v^2}{2E\sqrt{EG}} - \frac{E_vG_v}{2G\sqrt{EG}} + \frac{G_{uu}}{\sqrt{EG}} - \frac{G_u^2}{2G\sqrt{EG}} - \frac{E_uG_u}{2E\sqrt{EG}} \\
&= -2\sqrt{EG}\left(-\frac{E_{vv}}{4EG} + \frac{E_v^2}{4E^2G} + \frac{E_vG_v}{4EG^2} - \frac{G_{uu}}{4EG} + \frac{G_u^2}{4EG^2} + \frac{E_uG_u}{4E^2G}\right) \\
&= -2K\sqrt{EG}
\end{aligned}$$

and the result follows. □