3.3 Second fundamental form and Gaussian curvature

Example 3.3.8 (Catenoid). *Consider the surface obtained by rotating the catenary $x = f(z) = \cosh z$ in the xz-plane about the z axis which is called* **catenoid**. *The Gaussian curvature of catenoid is* :

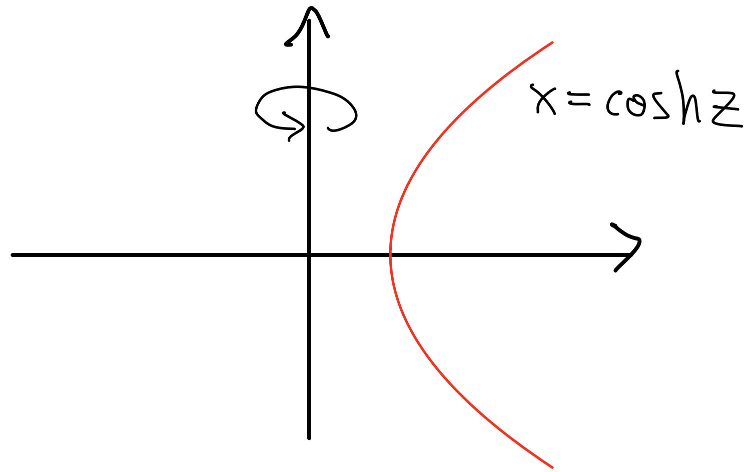
$$f(z) = \cosh z$$

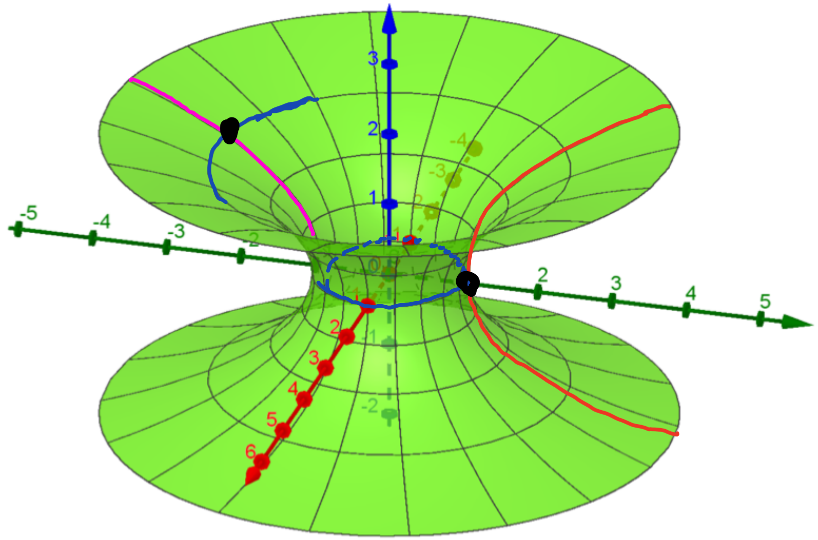
$$f'(z) = \sinh z$$

$$f''(z) = \cosh z$$

$$K(z) = -\frac{\cosh z}{\cosh z\,(1+\sinh^2 z)^2}$$

$$= -\frac{1}{\cosh^4 z} < 0$$





$$\boxed{K(z) = -\frac{f''}{f(1+f'^2)^2}.}$$

Example 3.3.9 (Torus). *Show that the Gaussian curvature of the torus obtained by rotating the arc length parametrized curve*

$s \in (0, 2\pi r)$

$$(x, z) = (\varphi(s), \psi(s)) = \left(R + r\sin\frac{s}{r}, r\cos\frac{s}{r}\right), \quad s \in \cancel{(0, 2\pi)}$$

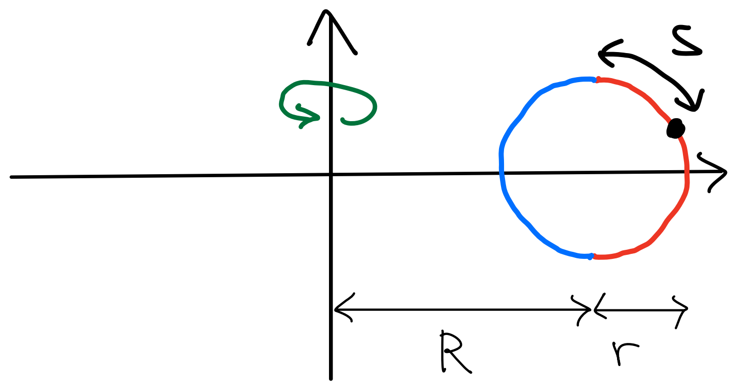
about the z-axis is

$$K = \frac{\sin\frac{s}{r}}{r(R + r\sin\frac{s}{r})}.$$

$$\boxed{K(s) = -\frac{\varphi''}{\varphi}.}$$

$$\varphi(s) = R + r\sin\frac{s}{r}$$

$$\varphi'(s) = \cos\frac{s}{r} \qquad \varphi''(s) = -\frac{1}{r}\sin\frac{s}{r}$$



$$K = -\frac{\varphi''}{\varphi}$$

$$= -\frac{-\frac{1}{r}\sin\frac{s}{r}}{R + r\sin\frac{s}{r}}$$

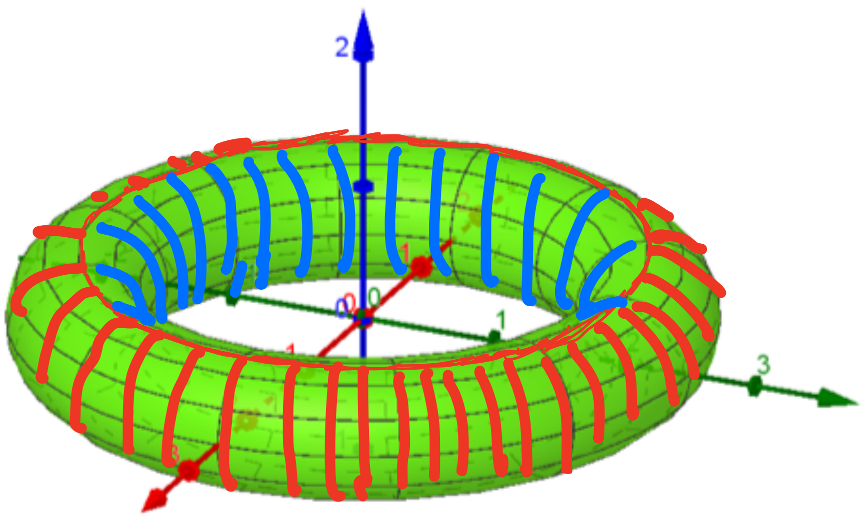
$$= \frac{\sin\frac{s}{r}}{r(R + r\sin\frac{s}{r})}$$

$K \geq 0$

"outside"

$K \leq 0$

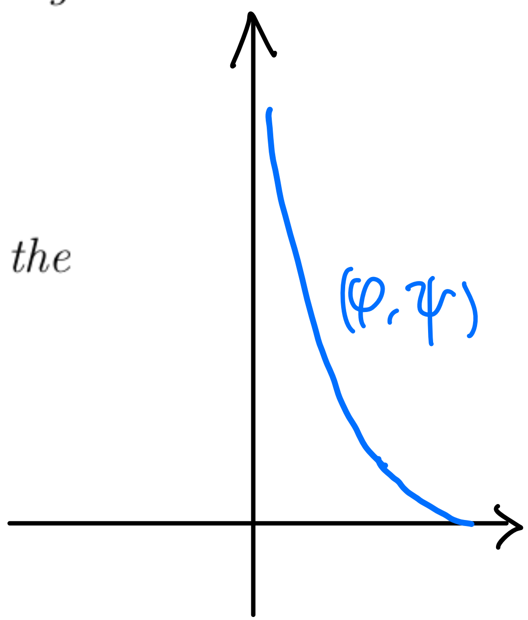
"inside"



Example 3.3.10 (Pseudosphere). *Consider the surface obtained by rotating the tractrix (Example 2.2.10)*

$$(x, z) = (\varphi(t), \psi(t)) = (\operatorname{sech} t, t - \tanh t)\,, \quad t > 0$$

about the z-axis. This surface is called the **pseudosphere**. *Show that the pseudosphere has constant Gaussian curvature equal to -1.*



Method 1 : $$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$

Method 2: Use arclength parametrization (2.2.10)

$$(\varphi(s), \psi(s)) = \left(e^{-s}, \ln\left(e^{s} + \sqrt{e^{2s} - 1}\right) - \sqrt{1 - e^{-2s}}\right)$$

$$K = -\frac{\varphi''(s)}{\varphi(s)} = -\frac{e^{-s}}{e^{-s}} = -1$$

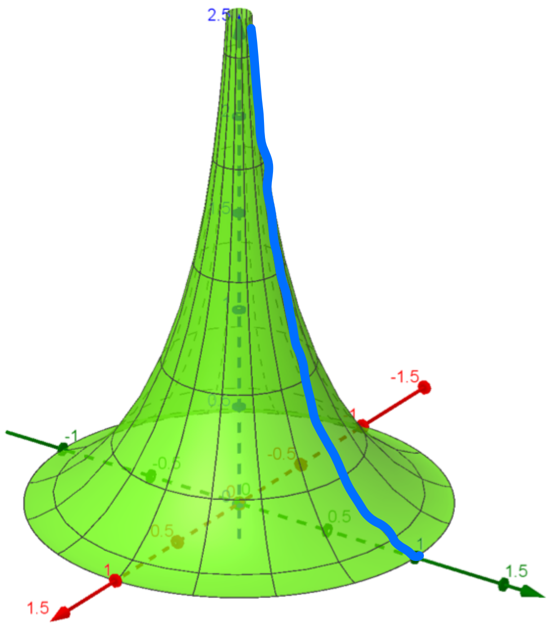


Figure 16: Pseudosphere

Theorem 3.3.11. *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface. Suppose* $F = 0$, *i.e., the first fundamental form of* $\mathbf{x}(u,v)$ *is*

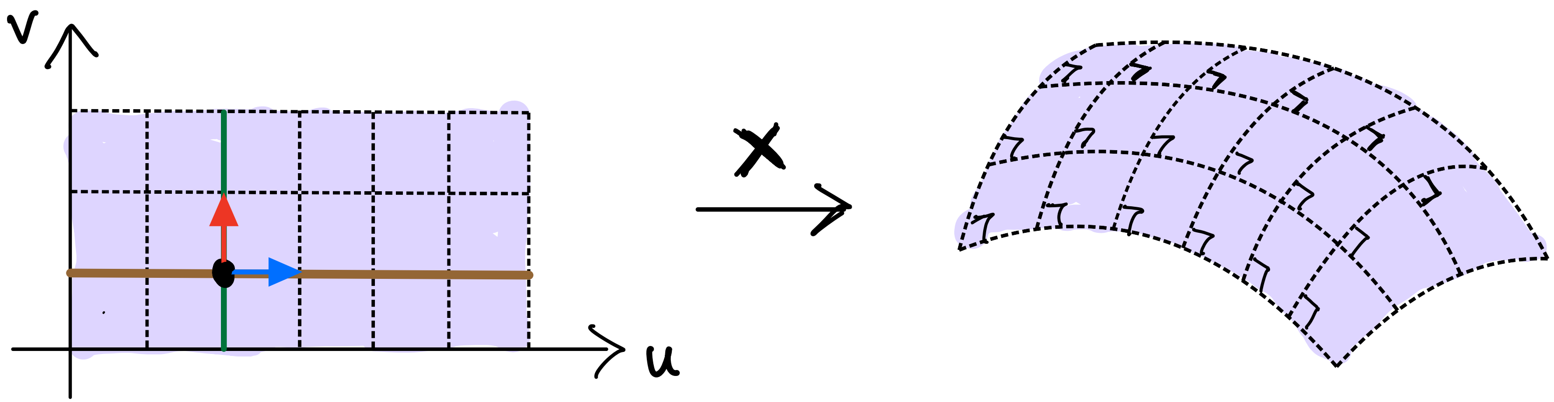
$$I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$

Then the Gaussian curvature of $\mathbf{x}(u,v)$ *is*

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Pf See Theorem 3.5.4

$F=0 \Rightarrow x_u \perp x_v$



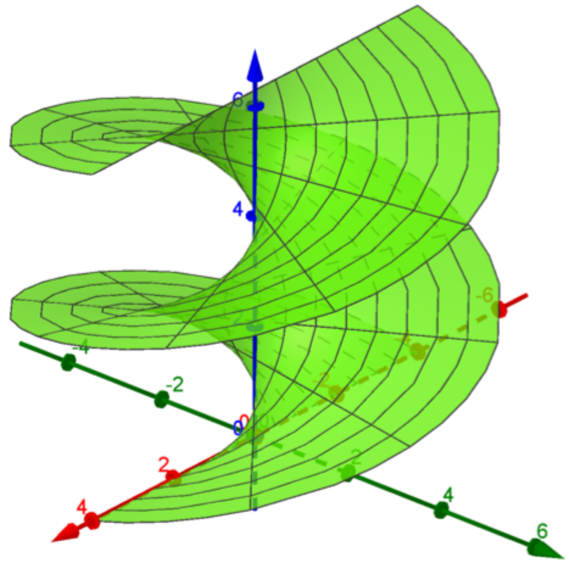
Example 3.3.12 (Helicoid). *Show that the Gaussian curvature of the* **helicoid** *parametrized by*

$$\mathbf{x}(u,\theta) = (u\cos\theta, u\sin\theta, \theta), \ \ u,\theta \in \mathbb{R},$$

is

$$K = -\frac{1}{(1+u^2)^2}.$$

$$X_u = (\cos\theta, \sin\theta, 0) \qquad X_\theta = (-u\sin\theta, u\cos\theta, 1)$$



$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1+u^2 \end{bmatrix} \qquad E=1 \quad F=0 \quad G=1+u^2$$

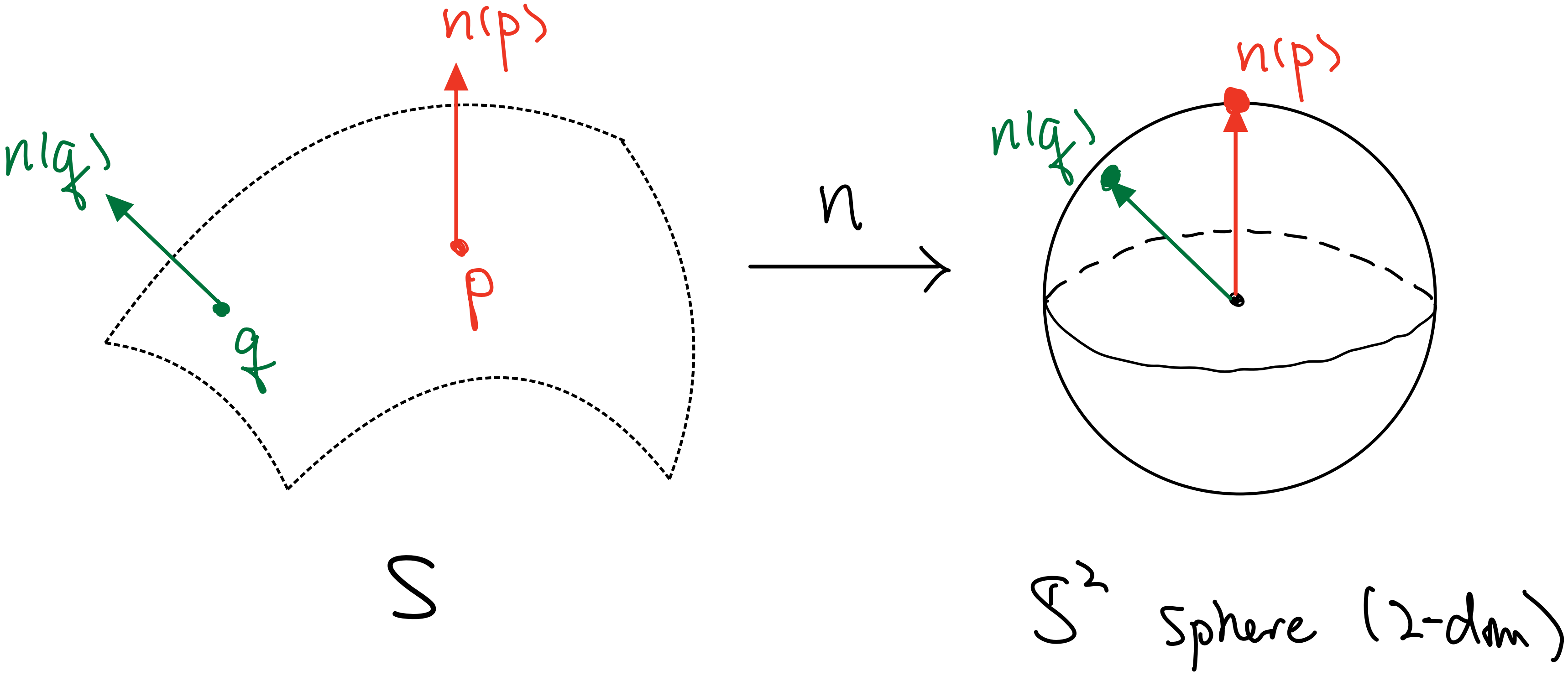
$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

$$= -\frac{1}{2\sqrt{1+u^2}}\left[0 + \left(\frac{2u}{\sqrt{1+u^2}}\right)_u\right]$$

$$= -\frac{1}{2\sqrt{1+u^2}} \cdot \frac{2\sqrt{1+u^2} - 2u\frac{u}{\sqrt{1+u^2}}}{1+u^2} = -\frac{1}{(1+u^2)^2}$$

3.4 Gauss map and its differential

Definition 3.4.1 (Gauss map). *Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u, v)$. For each $p = \mathbf{x}(u, v)$, we associate the unit normal vector $\mathbf{n}(p)$ to p. This defines a map $\mathbf{n} : S \to S^2$ from the surface S to the unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and is called the* **Gauss map** *of S.*



Proposition 3.4.2. *Let S be a regular surface with regular parametrization $\mathbf{x}(u, v)$ and $\mathbf{n} : S \to \mathbb{S}^2$ be the Gauss map which sends a point $p \in S$ to the unit normal vector $\mathbf{n} = \mathbf{n}(p)$ which is a point on the unit sphere $\mathbb{S}^2$. Let $p \in S$ be any point on the surface S. Then the following statements hold.*

1. *The unit normal vector $\mathbf{n} = \mathbf{n}(p)$ to the surface S is a unit normal vector to the unit sphere $\mathbb{S}^2$ at $\mathbf{n}$.*

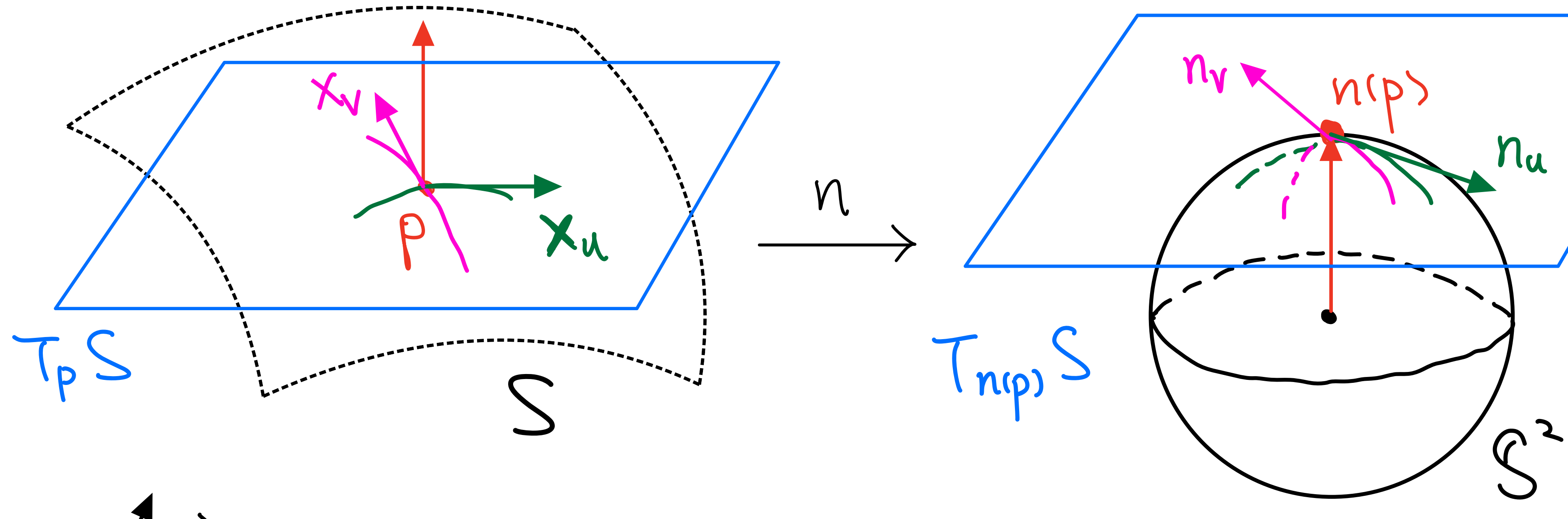
2. *The tangent space to the unit sphere $\mathbb{S}^2$ at $\mathbf{n} = \mathbf{n}(p)$ is equal to the tangent space to the surface S at p. In other words,*

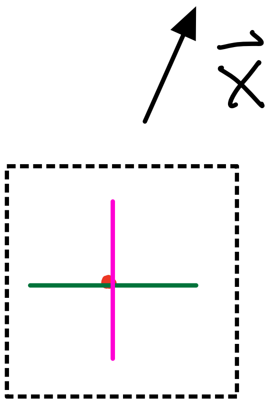
$$T_{\mathbf{n}}\mathbb{S}^2 = T_pS.$$

3. *The vectors $\mathbf{n}_u(p)$ and $\mathbf{n}_v(p)$ are tangent to S at p. In other words,*

$$\mathbf{n}_u, \mathbf{n}_v \in T_pS$$

   which means both $\mathbf{n}_u$, $\mathbf{n}_v$ can be written as linear combinations of $\mathbf{x}_u$ and $\mathbf{x}_v$.





- $T_pS = T_{n(p)}S^2 = \{\vec{v} \in \mathbb{R}^3 : \langle \vec{v}, n(p) \rangle = 0\}$
- $\|n\| \equiv 1 \Rightarrow \langle n, n \rangle \equiv 1 \Rightarrow \langle n, n \rangle_u = 0$

$\langle n_u, n \rangle + \langle n, n_u \rangle = 0 \Rightarrow \langle n_u, n \rangle = 0$

Hence $n_u, n_v \in T_{n(p)}S^2 = T_pS$

Later: find a, b, c, d such that $\begin{cases} n_u = a x_u + b x_v \\ n_v = c x_u + d x_v \end{cases}$

Theorem 3.4.3. *Let $\mathbf{x}(u,v)$ be a regular parametrized surface and $\mathbf{n}(u,v)$ be the unit normal vector at $\mathbf{x}(u,v)$. Then*

$$\mathbf{n}_u \times \mathbf{n}_v = K \mathbf{x}_u \times \mathbf{x}_v$$

where K is the Gaussian curvature of the surface.

Pf $n_u, n_v \perp x_u \times x_v$

$\Rightarrow n_u \times n_v = c\, x_u \times x_v$

$$\det(\mathrm{II}) = \begin{vmatrix} -\langle x_u, n_u\rangle & -\langle x_u, n_v\rangle \\ -\langle x_v, n_u\rangle & -\langle x_v, n_v\rangle \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} \langle x_u, n_u\rangle & \langle x_u, n_v\rangle \\ \langle x_v, n_u\rangle & \langle x_v, n_v\rangle \end{vmatrix}$$

$$= \langle x_u \times x_v, n_u \times n_v\rangle$$

$$= c\,\langle x_u \times x_v, x_u \times x_v\rangle$$

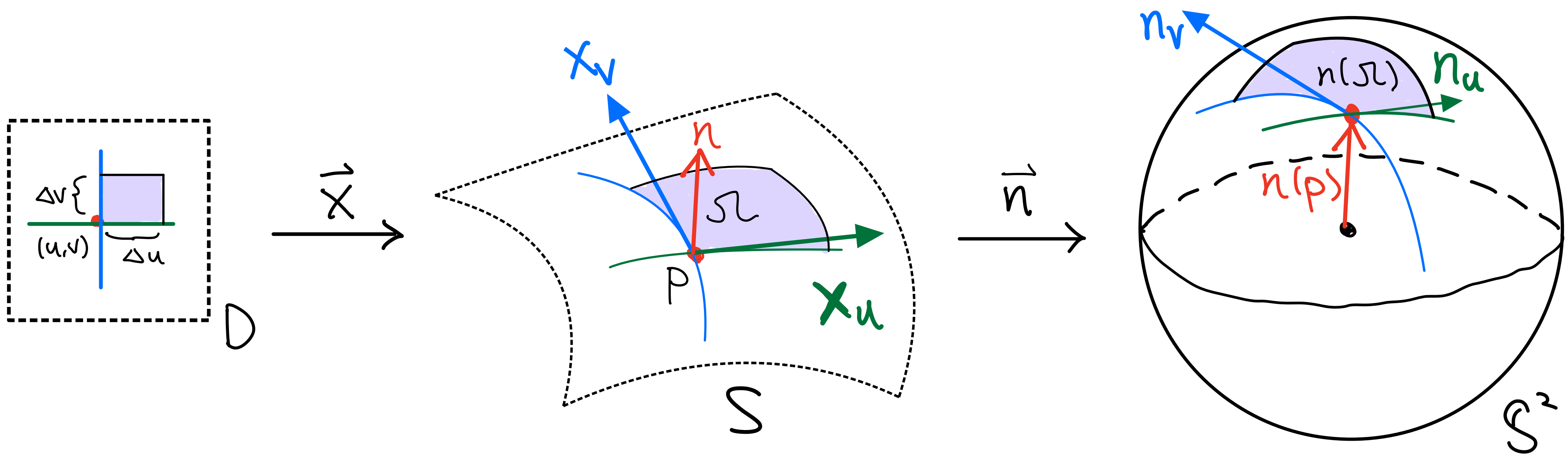
$$= c \det \mathrm{I}$$

Hence $c = \dfrac{\det \mathrm{II}}{\det \mathrm{I}} = K$

Formula

$$\begin{vmatrix} \langle a, c\rangle & \langle a, d\rangle \\ \langle b, c\rangle & \langle b, d\rangle \end{vmatrix} = \langle a \times b, c \times d\rangle$$

Gaussian Curvature as Area ratio



Define $\Omega = \{\mathbf{x}(s,t) : u < s < u + \Delta u, v < t < \Delta v\} \subset S$

ΔA = Area of Ω

$$\Delta\sigma = \text{signed area of } n(\Omega) = \begin{cases} \text{Area of } n(\Omega) & \text{if } n_u, n_v, n \text{ satisfy right hand rule} \\ -\text{Area of } n(\Omega) & \text{if } \underbrace{n_u, n_v, n \text{ satisfy left hand rule}}_{\text{orientation}} \end{cases}$$

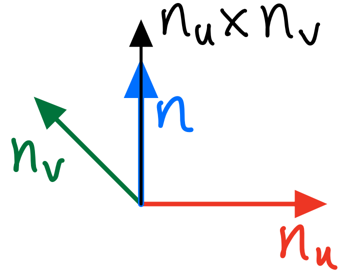
$$\Delta A \approx \| \Delta u\, X_u \times \Delta v\, X_v \| = \| X_u \times X_v \| \Delta u \Delta v$$

$$|\Delta\sigma| \approx \| \Delta u\, n_u \times \Delta v\, n_v \| = \| n_u \times n_v \| \Delta u\, \Delta v = \| K\, X_u \times X_v \| \Delta u \Delta v = |K|\, \Delta A$$

$\Delta\sigma \approx K \Delta A$ or $-K\Delta A$?

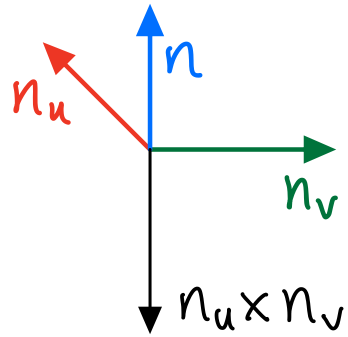
sign of $\Delta\sigma$ = sign of $\langle n_u \times n_v, n \rangle$

$$
\begin{aligned}
&\langle n_u \times n_v, n \rangle \\
&= \left\langle K X_u \times X_v, \frac{X_u \times X_v}{\|X_u \times X_v\|} \right\rangle \\
&= K \frac{\|X_u \times X_v\|^2}{\|X_u \times X_v\|} \\
&= K \|X_u \times X_v\| \quad \text{same sign as } K
\end{aligned}
$$



n_u, n_v, n satisfy right hand rule

$\langle n_u \times n_v, n \rangle > 0$



n_u, n_v, n satisfy left hand rule

$\langle n_u \times n_v, n \rangle < 0$

Hence $\Delta\sigma \approx K\Delta A$, $\dfrac{\Delta\sigma}{\Delta A} \approx K$ for small $\Delta u, \Delta v$

Proposition 3.4.4. *Let S be a regular surface with parametrization $\mathbf{x}(u,v)$, $(u,v) \in D$. Let A and σ be the signed surface area function on S and S^2 respectively. Then we have*

$$\frac{d\sigma}{dA} = K$$

where K is the Gaussian curvature.

<u>Rmk</u> $\dfrac{d\sigma}{dA} = \lim_{\Delta A \to 0} \dfrac{\Delta\sigma}{\Delta A}$

eg