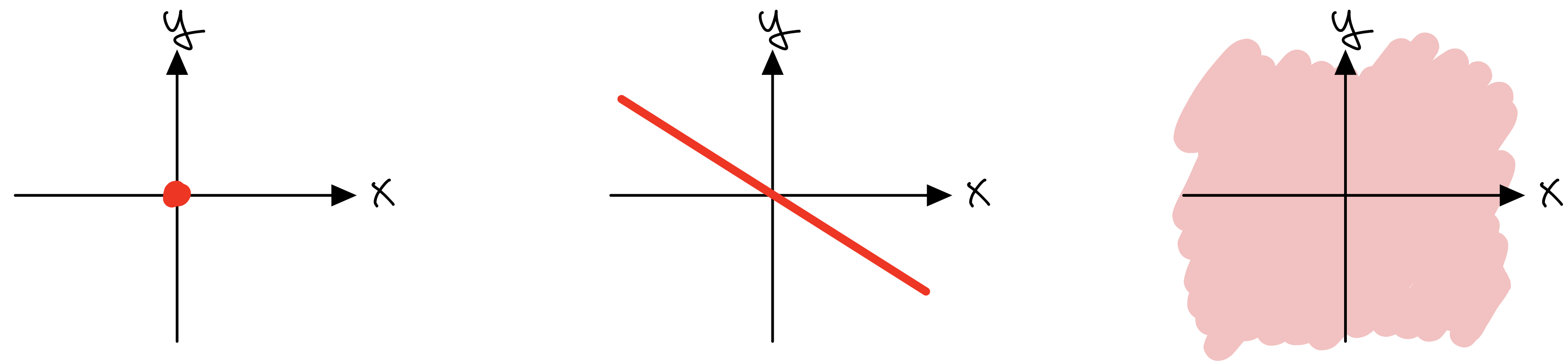
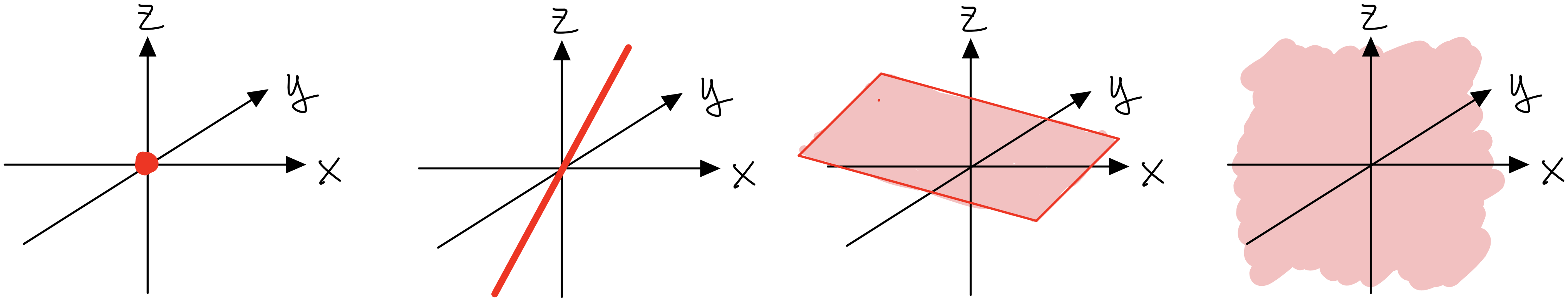
Definition 1.3.18 (Vector subspace). *We say that a subset $V \subset \mathbb{R}^m$ is a* **vector subspace** *of $\mathbb{R}^m$ if V contains the zero vector $\mathbf{0}$ and for any $\mathbf{u}, \mathbf{v} \in V$, $\alpha, \beta \in \mathbb{R}$, we have*

$$\alpha\mathbf{u} + \beta\mathbf{v} \in V.$$

eg vector subspaces of $\mathbb{R}^2$



eg vector subspaces of $\mathbb{R}^3$



Definition 1.3.19 (Linearly independent vectors and spanning set). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset V$ be a set of vectors in V.*

1. *We say that E is* **linearly independent** *if*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

   implies $c_1 = c_2 = \cdots = c_k = 0$.

2. *We say that E* **spans** *V if for any $\mathbf{v} \in V$, there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that*

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k.$$

Definition 1.3.23 (Basis). *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ be a set of vectors in V. We say that E constitutes a* **basis** *for V if*

1. *E is linearly independent, and*

2. *E spans V.*

Definition 1.3.28 (Dimension). *Let V be a vector subspace of $\mathbb{R}^m$. The* **dimension** *of V is the number of vectors in a basis for V and is denoted by* $\dim(V)$.

Theorem 1.3.25. *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ be a set of vectors in V. Then the following conditions are equivalent.*

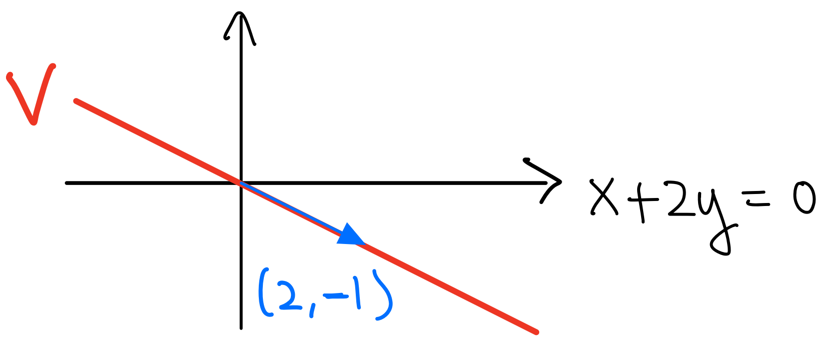
1. *E constitutes a basis for V.*
2. *For any $\mathbf{v} \in V$, there exists unique $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n.$$

- If V is a line ($\dim V = 1$) Any basis of V consists of 1 vector

  $E = \{v_1\}$, where $v_1 \in V$, $v_1 \neq 0$

eg



$E = \{(2,-1)\}$ is a basis

For any $\vec{v} \in V$, there exist a unique α such that $\vec{v} = \alpha(2,-1)$

eg $(6,-3) = 3(2,-1)$

$(-4,2) = -2(2,-1)$

$(0,0) = 0(2,-1)$

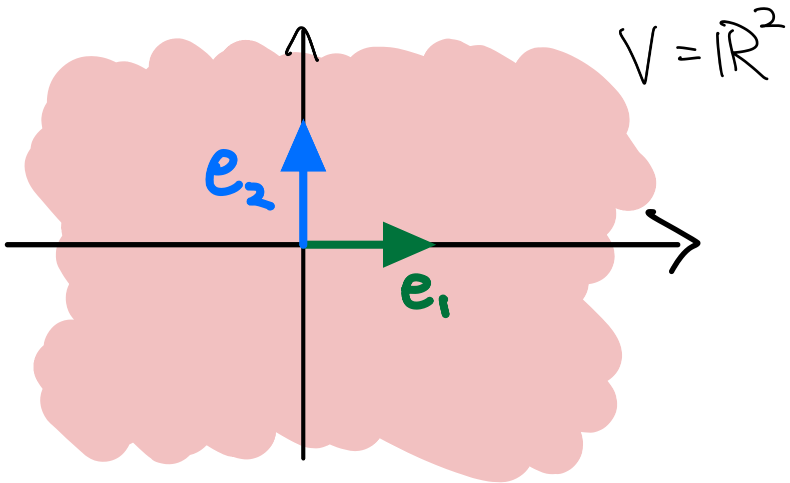
Theorem 1.3.25. *Let $V \subset \mathbb{R}^m$ be a vector subspace and $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ be a set of vectors in V. Then the following conditions are equivalent.*

1. *E constitutes a basis for V.*
2. *For any $\mathbf{v} \in V$, there exists unique $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that*

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

- If V is a plane ($\dim V = 2$) Any basis of V consists of 2 vectors
  $E = \{v_1, v_2\}$, where $v_1, v_2 \in V$ are non-zero and not in same/opposite direction

eg



$E = \{e_1, e_2\}$ is a basis

where $e_1 = (1,0)$ $e_2 = (0,1)$

For any $\vec{v} \in V$, there exist a unique α, β

such that $\vec{v} = \alpha e_1 + \beta e_2$

eg $(2,1) = 2e_1 + 1e_2$

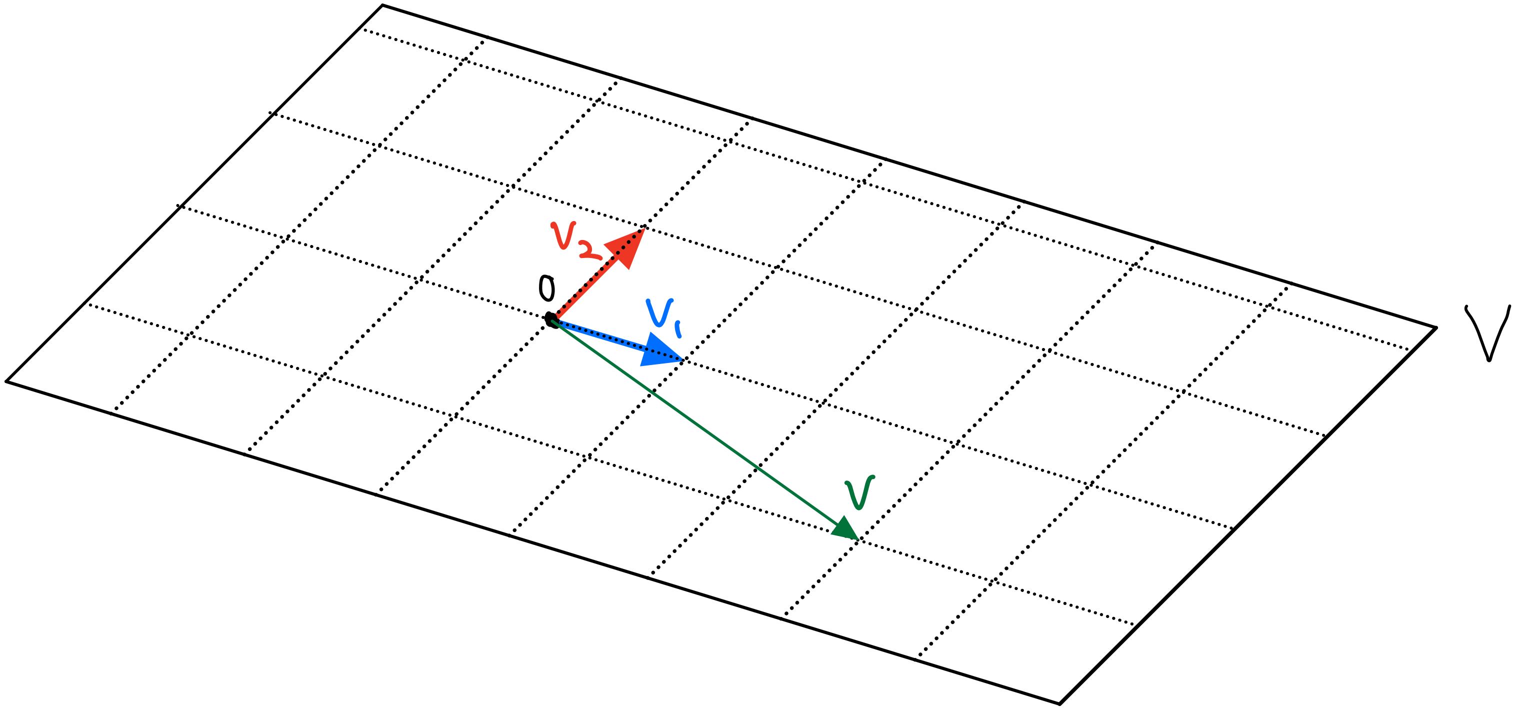
In general, for any $(x,y) \in V$

$(x,y) = x e_1 + y e_2$

eg $\quad V = \{(x,y,z) \in \mathbb{R}^3 : x - y + z = 0\} \subseteq \mathbb{R}^3$

$v_1 = (1,0,-1)$, $v_2 = (0,1,1) \in V$ not in same/opposite direction

$E = \{v_1, v_2\}$ is a basis of V



For any $\vec{v} \in V$, there exist a unique α, β such that $\vec{v} = \alpha v_1 + \beta v_2$

For example $v = (3,-1,-4) \in V$, $\quad v = 3v_1 + (-1)v_2$

Theorem 1.3.33. *The following conditions for $n \times n$ matrix A are equivalent.*

1. $\det(A) \neq 0$
2. *A is invertible, that is, the inverse A^{-1} of A exists.*
3. *For any n column vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for $\mathbf{x}$.*
4. *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has no nontrivial solution, that is, solution for which $\mathbf{x} \neq \mathbf{0}$.*
5. *The column vectors of A constitute a basis for $\mathbb{R}^m$.*

eg 1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\det A = -2 \neq 0$ A is invertible

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has only zero solution

$\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$

eg 2 $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\det B = 0$ B is not invertible

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has non-trivial solution for example, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ is not a basis for $\mathbb{R}^2$

1.7 Some transcendental functions

1. **Exponential function**:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ \textit{for } x \in \mathbb{R}$$

2. **Trigonometric functions**: *There are 6 trigonometric functions which are defined as follows.*

Cosine: $$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \ \textit{for } x \in \mathbb{R}$$

Sine: $$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ \textit{for } x \in \mathbb{R}$$

Tangent: $$\tan x = \frac{\sin x}{\cos x} \ \textit{for } x \neq \frac{(2k+1)\pi}{2}, \ k \in \mathbb{Z}$$

Cotangent: $$\cot x = \frac{\cos x}{\sin x} \ \textit{for } x \neq k\pi, \ k \in \mathbb{Z}$$

Secant: $$\sec x = \frac{1}{\cos x} \ \textit{for } x \neq \frac{(2k+1)\pi}{2}, \ k \in \mathbb{Z}$$

Cosecant: $$\csc x = \frac{1}{\sin x} \ \textit{for } x \neq k\pi, \ k \in \mathbb{Z}$$

3. **Hyperbolic functions**: *There are 6 hyperbolic functions which are defined as follows.*

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \textit{ for } x \in \mathbb{R}$$

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \textit{ for } x \in \mathbb{R}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} \textit{ for } x \in \mathbb{R}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} \textit{ for } x \neq 0$$

Hyperbolic secant:

$$\mathrm{sech} x = \frac{1}{\cosh x} \textit{ for } x \in \mathbb{R}$$

Hyperbolic cosecant:

$$\mathrm{csch} x = \frac{1}{\sinh x} \textit{ for } x \neq 0$$

Theorem 1.7.2. *The exponential function satisfies*

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

for any $x \in \mathbb{R}$.

Definition 1.7.3 (Logarithmic function). *The* **logarithmic function** *is the function* $\ln : \mathbb{R}^+ \to \mathbb{R}$ *defined for* $x > 0$ *by*

$$y = \ln x \textit{ if } e^y = x.$$

In other words, $\ln x$ *is the inverse function of the exponential function.*

Proposition 1.7.4 (Identities for transcendental functions).

1. *Exponential function:*
   (a) $e^{x+y} = e^x e^y$
   (b) $e^{x-y} = \dfrac{e^x}{e^y}$
   (c) $e^{kx} = (e^x)^k$ *for* $k \in \mathbb{Z}$
2. *Logarithmic function:*
   (a) $\ln(xy) = \ln x + \ln y$
   (b) $\ln \dfrac{x}{y} = \ln x - \ln y$
   (c) $\ln(x^k) = k \ln x$ *for* $k \in \mathbb{Z}$
3. *Trigonometric identities:*
   (a) $\cos^2 x + \sin^2 x = 1;\quad \sec^2 x - \tan^2 x = 1;\quad \csc^2 x - \cot^2 x = 1$
   (b) $\cos(-x) = \cos x;\quad \sin(-x) = -\sin x;\quad \tan(-x) = -\tan x$
   (c) $\cos(x+y) = \cos x \cos y - \sin x \sin y;$
   $\sin(x+y) = \sin x \cos y + \cos x \sin y;$
   $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
   (d) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x;$
   $\sin 2x = 2 \sin x \cos x;$
   $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
4. *Hyperbolic identities:*
   (a) $\cosh^2 x - \sinh^2 x = 1;\quad \text{sech}^2 x + \tanh^2 x = 1;\quad \coth^2 x - \text{csch}^2 x = 1$
   (b) $\cosh(-x) = \cosh x;\quad \sinh(-x) = -\sinh x;\quad \tanh(-x) = -\tanh x$
   (c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y;$
   $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y;$
   $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
   (d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x;$
   $\sinh 2x = 2 \sinh x \cosh x;$
   $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Proposition 1.7.5 (Derivatives of transcendental functions).

1. *Exponential function:*

$$\frac{d}{dx}e^x = e^x$$

2. *Logarithmic function:*

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

3. *Trigonometric functions:*

$$\frac{d}{dx}\cos x = -\sin x; \qquad \frac{d}{dx}\sin x = \cos x;$$
$$\frac{d}{dx}\tan x = \sec^2 x; \qquad \frac{d}{dx}\cot x = -\csc^2 x;$$
$$\frac{d}{dx}\sec x = \sec x \tan x; \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

4. *Inverse trigonometric functions*[7]*:*

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}};$$
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}};$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

5. *Hyperbolic functions:*

$$\frac{d}{dx}\cosh x = \sinh x; \qquad \frac{d}{dx}\sinh x = \cosh x;$$
$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x; \qquad \frac{d}{dx}\coth x = -\mathrm{csch}^2 x;$$
$$\frac{d}{dx}\mathrm{sech} x = -\mathrm{sech} x \tanh x; \qquad \frac{d}{dx}\mathrm{csch} x = -\mathrm{csch} x \coth x$$

6. *Inverse hyperbolic functions*[8]*:*

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}};$$
$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}};$$
$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

Proposition 1.7.6 (Integrals of transcendental functions)**.**

1. *Exponential function:*

$$\int e^x dx = e^x + C$$

2. *Logarithmic function:*

$$\int \frac{1}{x} dx = \ln|x| + C$$

3. *Trigonometric functions:*

$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C;$$

$$\int \tan x dx = \ln \sec x; \qquad \int \cot x = \ln \sin x + C;$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C; \quad \int \csc x dx = \ln|\csc x - \cot x| + C$$

4. *Hyperbolic functions:*

$$\int \cosh x dx = \sinh x + C; \qquad \int \sinh x dx = \cosh x + C;$$

$$\int \tanh x dx = \ln \cosh x; \qquad \int \coth x = \ln \sinh x + C;$$

$$\int \text{sech} x dx = \tan^{-1} \sinh x + C; \quad \int \text{csch} x dx = \ln|\text{csch} x - \coth x| + C$$

2 Curves

2.1 Regular parametrized curves

Definition 2.1.1 (Regular parametrized curves). *A* **regular parametrized curve** *is a differentiable function* $\mathbf{r} : (a, b) \to \mathbb{R}^n$, $n = 2$ *or* 3, *such that* $\mathbf{r}'(t) \neq \mathbf{0}$ *for any* $t \in (a, b)$.

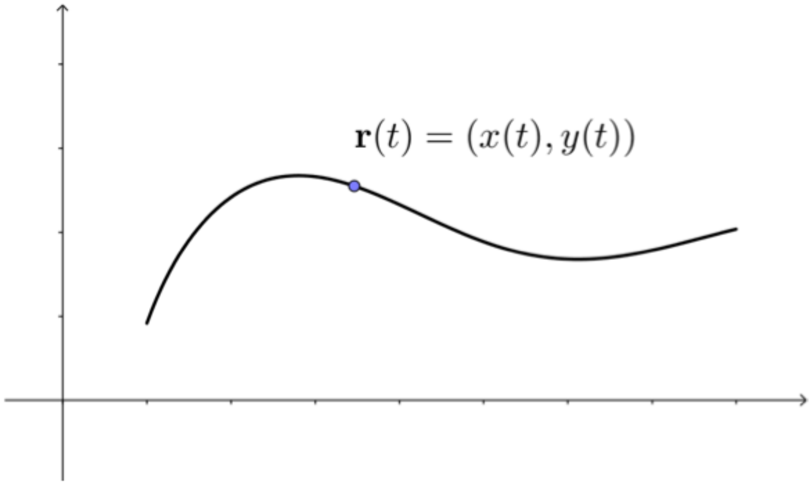
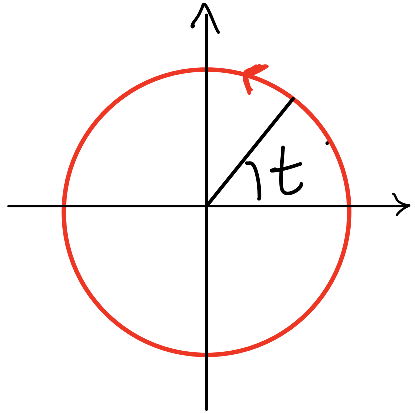


Figure 1: Regular parametrized curve

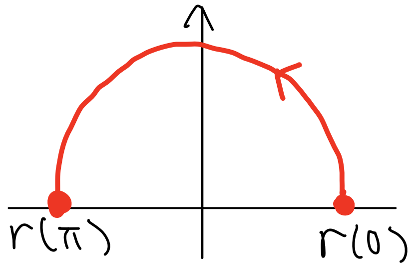
Rmk
- t is called parameter
- regular means $r'(t) \neq \vec{0}$
- We will also consider curve defined on closed interval and unbounded interval

- $r(t) = (\cos t, \sin t)$ $t \in (-\infty, \infty)$

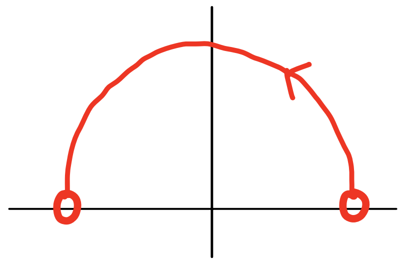


loop around infinity time

- $r(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$



- $r(t) = (\cos t, \sin t)$, $0 < t < \pi$



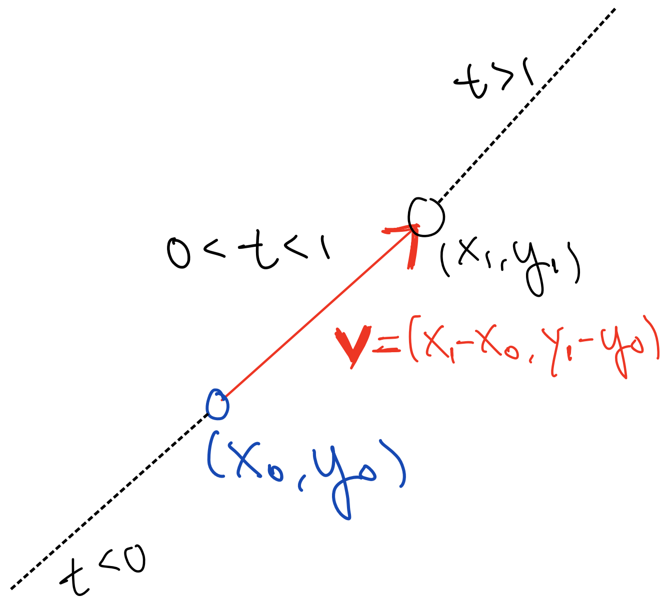
1. *Straight line: Let (x_0, y_0) and (x_1, y_1) be two points on $\mathbb{R}^2$. The function*

$$\mathbf{r}(t) = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \text{ for } 0 < t < 1$$

$$\vec{r}(t) = (x_0 - tx_0 + tx_1, y_0 - ty_0 + ty_1)$$

$$= (x_0, y_0) + (t(x_1 - x_0), t(y_1 - y_0))$$

$$= \underbrace{(x_0, y_0)}_{\text{"Starting point"}} + t\underbrace{(x_1 - x_0, y_1 - y_0)}_{\text{vector from } (x_0, y_0) \text{ to } (x_1, y_1)}$$



2. *Circle: Let $r > 0$ be a positive real number. The function*

$$\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta), \text{ for } 0 < \theta < 2\pi$$

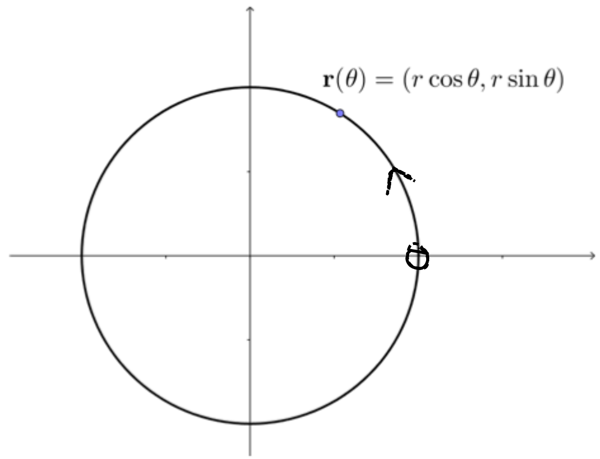


Figure 3: Circle

3. *Cycloid: The function*

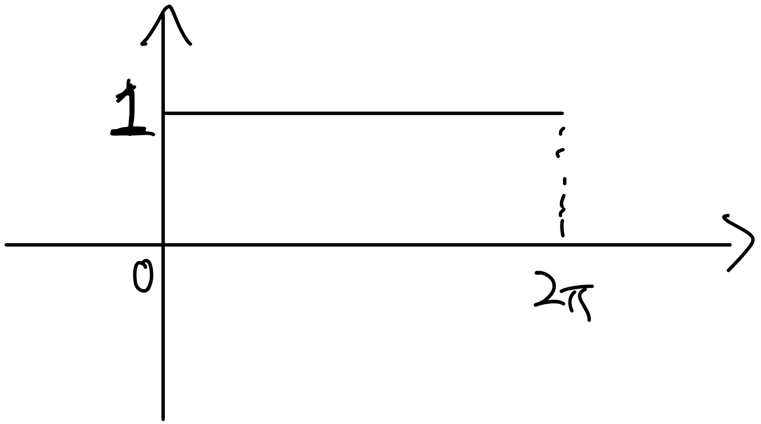
$$\mathbf{r}(\theta) = (\theta - \sin\theta, 1 - \cos\theta), \ \textit{for } 0 < \theta < 2\pi$$

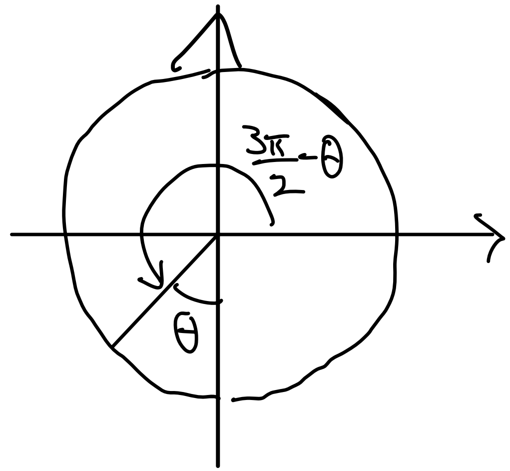
$$= \underbrace{(\theta, 1)}_{r_1} + \underbrace{(-\sin\theta, -\cos\theta)}_{r_2}$$

$r_2(\theta)$

$= (-\sin\theta, -\cos\theta)$

$= \left(\cos\left(\frac{3\pi}{2} - \theta\right), \sin\left(\frac{3\pi}{2} - \theta\right)\right)$





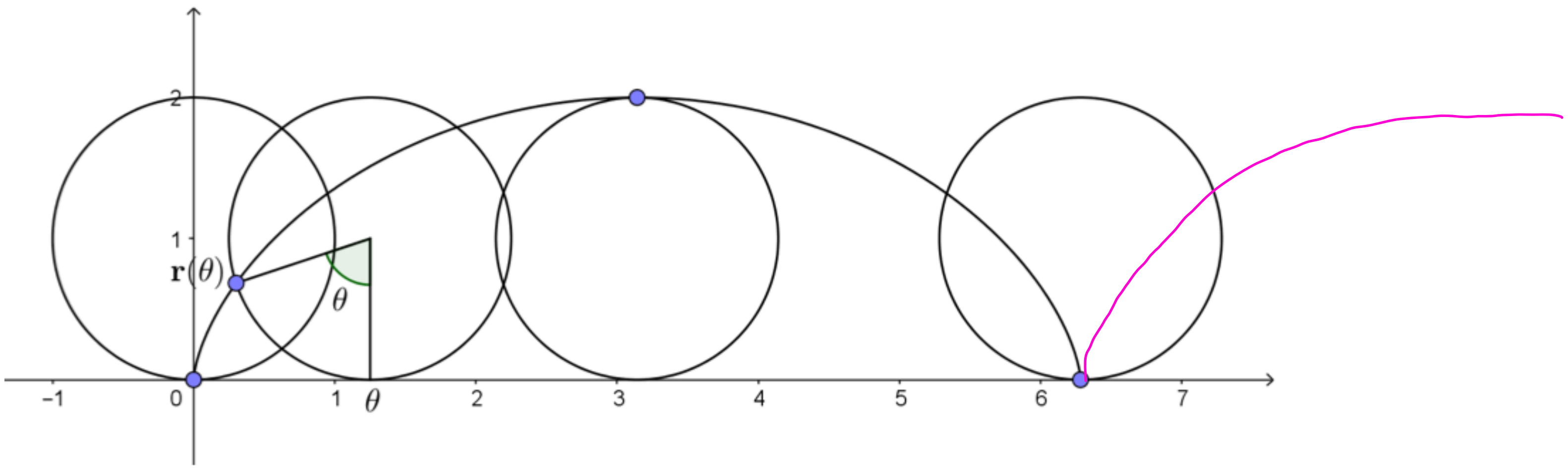


Figure 4: Cycloid

Q Consider $r(\theta) = (\theta - \sin\theta,\ 1 - \cos\theta) \qquad 0 < \theta < 4\pi$

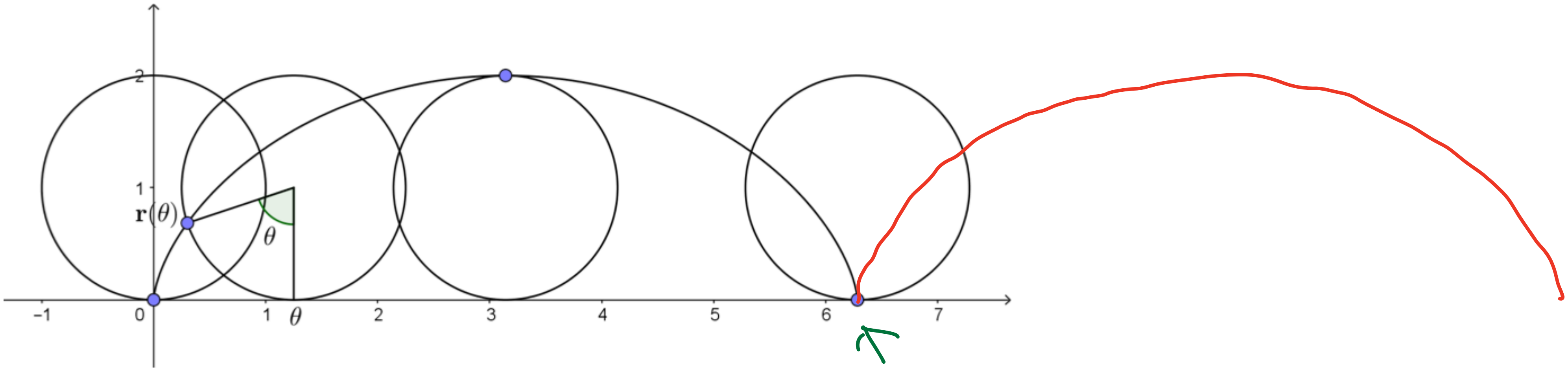


Figure 4: Cycloid

sharp corner?

differentiable?

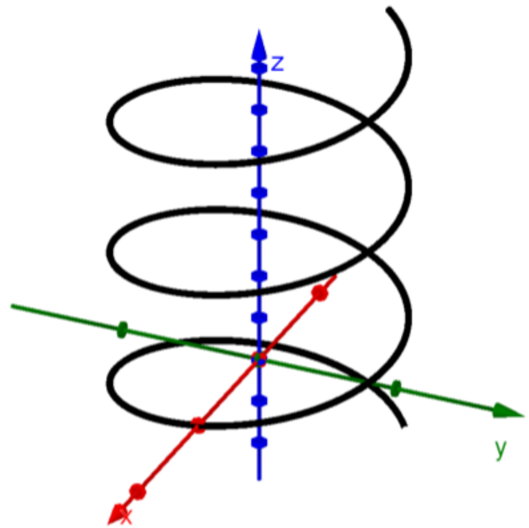
Note $r'(\theta) = (1 - \cos\theta,\ \sin\theta)$

$r'(2\pi) = (1-1,\ 0) = (0,0) \leftarrow$ not regular

4. Helix: The function $= (a\cos\theta, a\sin\theta, 0) + (0, 0, b\theta)$

$$\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta), \quad \textit{for } \theta \in \mathbb{R}$$

defines a curve which is called a **helix**.



$x \quad y$

Example 2.1.3. *Let* $\mathbf{r}(t) = (t^2, t^3)$. *Then* $\mathbf{r}'(t) = (2t, 3t^2)$ *and* $\mathbf{r}'(0) = (0, 0)$. *Therefore* $\mathbf{r}(t)$ *is not regular at* $t = 0$.

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases} \Longrightarrow \quad x^3 = y^2$$

$$y = \pm x^{\frac{3}{2}}$$