MMAT5390: Mathematical Image Processing

Assignment 5 Solutions

1. (a) Please check that $\dfrac{\partial \vec{f}^T\vec{a}}{\partial \vec{f}} = \vec{a}$, $\dfrac{\partial \vec{b}^T\vec{f}}{\partial \vec{f}} = \vec{b}$ and $\dfrac{\partial \vec{f}^T A\vec{f}}{\partial \vec{f}} = (A + A^T)\vec{f}$.

   We know the minimizer must satisfy

$$\mathcal{D} = \frac{\partial}{\partial \vec{f}}[\vec{f}^T L_1^T L_1 \vec{f} + \vec{f}^T L_2^T L_2 \vec{f} + \lambda(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f})] = 0$$

   where λ is the Lagrange multiplier. Therefore,

$$\begin{aligned}
&\mathcal{D} = 0 \\
&\Rightarrow 2(L_1^T L_1 + L_2^T L_2)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^T D\vec{f}) = 0 \\
&\Rightarrow (\lambda D^T D + L_1^T L_1 + L_2^T L_2)\vec{f} = \lambda D^T \vec{g}.
\end{aligned}$$

   (b) Since L_1, L_2 are also block-circulant, they are also diagonalizable by W. Denote $W^{-1}DW$ by Λ_D, $W^{-1}L_1W$ by Λ_{L_1} and $W^{-1}L_2W$ by Λ_{L_2}. Then

$$\begin{aligned}
\lambda D^T D + L_1^T L_1 + L_2^T L_2 &= \lambda D^* D + L_1^* L_1 + L_2^* L_2 \\
&= \lambda(W\Lambda_D W^{-1})^*(W\Lambda_D W^{-1}) + (W\Lambda_{L_1}W^{-1})^*(W\Lambda_{L_1}W^{-1}) \\
&\quad + (W\Lambda_{L_2}W^{-1})^*(W\Lambda_{L_2}W^{-1}) \\
&= \lambda W\Lambda_D^*\Lambda_D W^{-1} + W\Lambda_{L_1}^*\Lambda_{L_1}W^{-1} + W\Lambda_{L_2}^*\Lambda_{L_2}W^{-1}
\end{aligned}$$

   and $\lambda D^T = \lambda D^* = \lambda(W\Lambda_D W^{-1})^* = \lambda W\Lambda_D^* W^{-1}$. Hence

$$\begin{aligned}
(\lambda W\Lambda_D^*\Lambda_D W^{-1} + W\Lambda_{L_1}^*\Lambda_{L_1}W^{-1} + W\Lambda_{L_2}^*\Lambda_{L_2}W^{-1})\mathcal{S}(f) &= \lambda W\Lambda_D^* W^{-1}\mathcal{S}(g), \\
(\lambda\Lambda_D^*\Lambda_D + \Lambda_{L_1}^*\Lambda_{L_1} + \Lambda_{L_2}^*\Lambda_{L_2})\mathcal{S}(DFT(f)) &= \lambda\Lambda_D^*\mathcal{S}(DFT(g))
\end{aligned}$$

   and thus

$$\begin{aligned}
DFT(f)(u,v) &= \frac{\lambda N^2\overline{DFT(h)(u,v)}}{N^4[\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2 + |DFT(q)(u,v)|^2]}DFT(g)(u,v) \\
&= \frac{1}{N^2}\frac{\lambda\overline{DFT(h)(u,v)}}{\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2 + |DFT(p)(u,v)|^2}DFT(g)(u,v).
\end{aligned}$$

2. (a) After applying mean filter,

$$\tilde{I} = \frac{1}{9}\begin{pmatrix} 24 & 25 & 24 & 23 \\ 25 & 28 & 25 & 24 \\ 24 & 25 & 28 & 25 \\ 23 & 24 & 25 & 24 \end{pmatrix}$$

   (b) After applying median filter,

$$\tilde{I} = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

   (c)

$$\tilde{I} = K * I = \begin{pmatrix} 2 & 0 & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & 0 \\ -2 & -\frac{1}{2} & 0 & 2 \end{pmatrix}$$

3. Note that

$$\begin{aligned}
I_{sharp}(0,0) &= I(0,0) - \Delta I(0,0) \\
4 &= 1 - (-4 + a + 0 + b + 0) \\
a + b &= 1 \\
I_{sharp}(0,2) &= I(0,2) - \Delta I(0,2) \\
-2 &= 0 - (0 + 0 + 0 + 0 + b) \\
b &= 2 \\
I_{sharp}(1,3) &= I(1,3) - \Delta I(1,3) \\
1 &= 0 - (0 + c + 0 + a + 0) \\
a + c &= -1
\end{aligned}$$

By solving the above equations, we have $a = -1, b = 2$ and $c = 0$.

4. (a) Suppose f is a minimizer of E. Then for any $v : \Omega \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\bigg|_{t=0} E(f + tv) \\
&= \int_\Omega \frac{\partial}{\partial t}\bigg|_{t=0} [(f + tv - g)^2 + \lambda\|\nabla(f + tv)\|_2^2]\, dx\, dy \\
&= \int_\Omega [2v(f - g) + \lambda \frac{\partial}{\partial t}\bigg|_{t=0} \langle \nabla f + t\nabla v, \nabla f + t\nabla v\rangle]\, dx\, dy \\
&= \int_\Omega [2v(f - g) + \lambda \frac{\partial}{\partial t}\bigg|_{t=0} (\|\nabla f\|_2^2 + 2t\langle \nabla f, \nabla v\rangle + t^2\|\nabla v\|_2^2)]\, dx\, dy \\
&= 2\int_\Omega [v(f - g) + \lambda\langle \nabla f, \nabla v\rangle]\, dx\, dy \\
&= 2\int_\Omega v[f - g - \lambda\nabla \cdot (\nabla f)] + 2\lambda \int_{\partial\Omega} v\langle \nabla f, \vec{n}\rangle\, ds. \quad (*)
\end{aligned}$$

Since v is arbitrarily chosen, f satisfies:

$$\begin{cases} -\lambda\nabla^2 f(x,y) + f(x,y) = g(x,y) & \text{if } (x,y) \in \Omega, \\ \langle \nabla f(x,y), \vec{n}(x,y)\rangle = 0 & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

(b) Suppose f satisfies the above PDE. Then for any $h : \Omega \to \mathbb{R}$,

$$\begin{aligned}
E(h) - E(f) &= \int_\Omega \left[(h - g)^2 - (f - g)^2 + \lambda\|\nabla h\|_2^2 - \lambda\|\nabla f\|_2^2\right]\, dx\, dy \\
&= \int_\Omega \left[h^2 - 2gh + g^2 - f^2 + 2fg - g^2 + \lambda\left(\|\nabla h - \nabla f\|_2^2 - 2\|\nabla f\|_2^2 + 2\langle \nabla h, \nabla f\rangle\right)\right]\, dx\, dy \\
&= \int_\Omega \left[(h - f)^2 + 2hf - 2f^2 - 2gh + 2fg + \lambda\left(\|\nabla h - \nabla f\|_2^2 + 2\langle \nabla f, \nabla h - \nabla f\rangle\right)\right]\, dx\, dy \\
&\geq 2\int_\Omega [(f - g)(h - f) + \lambda\langle \nabla f, \nabla h - \nabla f\rangle]\, dx\, dy \\
&= 2\int_\Omega \left[(f - g)(h - f) - \lambda(h - f)\nabla^2 f\right]\, dx\, dy + 2\int_{\partial\Omega} (h - f)\langle \nabla f, \vec{n}\rangle\, ds = 0.
\end{aligned}$$

Hence f minimizes E.

(c) From (*), a descent direction of E at f is given by $-v$ where:

$$v(x,y) = \begin{cases} -f(x,y) + g(x,y) + \lambda\nabla^2 f(x,y) & \text{if } (x,y) \in \Omega, \\ -\langle \nabla f(x,y), \vec{n}(x,y)\rangle & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

Hence given $f : \Omega \to \mathbb{R}$, the gradient descent scheme for minimizing E is given by:

$$\frac{\partial f(x,y;t)}{\partial t} = \begin{cases} -f(x,y) + g(x,y) + \lambda \nabla^2 f(x,y) & \text{if } (x,y) \in \Omega, \\ -\langle \nabla f(x,y), \vec{n}(x,y) \rangle & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

5. (Coding assignment, optional)

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if t >= n/(2^p) && t < (n+0.5)/(2^p)
    H(i, j) = sqrt(2)^p;
elseif t >= (n+0.5)/(2^p) && t < (n+1)/(2^p)
    H(i, j) = -sqrt(2)^p;
end
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img = img + G(i, j) * Ht(:, i) * H(j, :);
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