MMAT5390: Mathematical Image Processing

Assignment 3 Solutions

1.

$$\begin{aligned}
RHS &= \sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\widehat{g}(p,q)\widehat{w}(k-p,l-q)\\
&= \sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}g(m,n)e^{-2\pi j(\frac{mp}{M}+\frac{nq}{N})}\frac{1}{MN}\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}w(s,t)e^{-2\pi j(\frac{s(k-p)}{M}+\frac{t(l-q)}{N})}\\
&= \frac{1}{(MN)^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}g(m,n)\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}w(s,t)\sum_{p=0}^{M-1}e^{-2\pi j(\frac{p(m-s)}{M}+\frac{sk}{M})}\sum_{q=0}^{N-1}e^{-2\pi j(\frac{q(n-t)}{N}+\frac{tl}{N})}
\end{aligned}$$

Note that when we fix k, l, m, n, s, t, we have

$$\begin{aligned}
\sum_{p=0}^{M-1}e^{-2\pi j\left(\frac{p(m-s)}{M}+\frac{sk}{M}\right)} &= \sum_{p=0}^{M-1}e^{-2\pi j\left(\frac{p(m-s)}{M}\right)}e^{-2\pi j\left(\frac{sk}{M}\right)}\\
&= e^{-2\pi j\left(\frac{sk}{M}\right)}\sum_{p=0}^{M-1}e^{-2\pi j\left(\frac{p(m-s)}{M}\right)}\\
&= \begin{cases} M\cdot e^{-2\pi j\left(\frac{sk}{M}\right)} & \text{if } m=s,\\ e^{-2\pi j\left(\frac{sk}{M}\right)}\cdot\frac{1-\left(e^{-2\pi j\left(\frac{m-s}{M}\right)}\right)^M}{1-e^{-2\pi j\left(\frac{m-s}{M}\right)}}=0 & \text{otherwise}\end{cases}\\
&= M\cdot e^{-2\pi j\left(\frac{sk}{M}\right)}\mathbb{1}_{\{0\}}(m-s)
\end{aligned}$$

Similarly, we also have

$$\sum_{q=0}^{N-1}e^{-2\pi j(\frac{q(n-t)}{N}+\frac{tl}{N})} = N\cdot e^{-2\pi j\left(\frac{tl}{N}\right)}\mathbb{1}_{\{0\}}(n-t)$$

Combining the above we have

$$\begin{aligned}
RHS &= \frac{1}{(MN)^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}g(m,n)\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}w(s,t)M\cdot e^{-2\pi j\left(\frac{sk}{M}\right)}\mathbb{1}_{\{0\}}(m-s)N\cdot e^{-2\pi j\left(\frac{tl}{N}\right)}\mathbb{1}_{\{0\}}(n-t)\\
&= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}g(m,n)w(m,n)e^{-2\pi j\left(\frac{mk}{M}+\frac{nl}{N}\right)}\\
&= \widehat{g\odot w}(k,l)\\
&= \widehat{x}(k,l)\\
&= LHS
\end{aligned}$$

2. Note that $\tilde{g}(k,l) = g(k,l) \times e^{2\pi j \frac{m_0 k + n_0 l}{N}}$.

$$\begin{aligned} DFT(g)(m-m_0, n-n_0) &= \frac{1}{N^2} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi j \frac{k(m-m_0)+l(n-n_0)}{N}} \\ &= \frac{1}{N^2} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) e^{2\pi j \frac{km_0+ln_0}{N}} e^{-2\pi j \frac{km+ln}{N}} \\ &= \frac{1}{N^2} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \tilde{g}(k,l) e^{-2\pi j \frac{km+ln}{N}} \\ &= DFT(\tilde{g})(m,n) \end{aligned}$$