

MMAT5390 Mathematical Image Processing

Final Practice Solutions

Solutions prepared by TAs, for your reference only.

1. The Butterworth high-pass filter H with radius D_0 and order n is defined as

$$H(u, v) = \frac{1}{1 + (D_0/D(u, v))^n},$$

where $D(u, v) = u^2 + v^2$. Given an image $I = (I(m, n))_{0 \leq m, n \leq 2N}$ and $N > 100$, apply Butterworth high-pass filter on $DFT(I) = (\hat{I}(u, v))_{0 \leq u, v \leq 2N}$ then get $G(u, v)$. Suppose

$$G(3, 4) = \frac{1}{2} \hat{I}(3, 4) \text{ and } G(2N - 6, 8) = \frac{16}{17} \hat{I}(2N - 6, 8),$$

where $\hat{I}(3, 4) \neq 0$ and $\hat{I}(2N - 6, 8) \neq 0$. Find D_0 and n .

Solution: The given information implies

$$H(3, 4) = \frac{1}{2} \text{ and } H(2N - 6, 8) = \frac{16}{17}.$$

After centralization, we have $H(-6, 8) = H(2N - 6, 8) = \frac{16}{17}$ Hence

$$\begin{cases} \frac{25^n}{D_0^n + 25^n} = \frac{1}{2}, \\ \frac{100^n}{D_0^n + 100^n} = \frac{16}{17}, \end{cases} \text{ and thus } \begin{cases} 25^n = D_0^n, \\ 4^n \cdot 25^n = 16D_0^n. \end{cases}$$

Then $4^n = 16$. Hence $n = 2$ and then $D_0 = 25$.

2. Consider a 4×4 periodically extended image $I = (I(k, l))_{0 \leq k, l \leq 3}$ given by:

$$I = \begin{pmatrix} a & b & a & b \\ b & a & b & a \\ a & b & a & b \\ b & a & b & a \end{pmatrix},$$

where a and b are distinct positive numbers.

$$\text{Let } h = (h(k, l))_{0 \leq k, l \leq 3} = \frac{1}{8} \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ which is periodically extended.}$$

The Gaussian low-pass filter H of variance σ^2 is defined by:

$$H(u, v) = \exp\left(-\frac{u^2 + v^2}{\sigma^2}\right).$$

Let $I_1(u, v) = H_1(u, v)DFT(I)(u, v)$, where the H_1 is the Gaussian low-pass filter of variance ab .

Suppose $I_1(2, 2) = e^{-\frac{1}{4}}DFT(I)(2, 2)$ and $h * I(2, 1) = 6$. Find a and b .

Solution: Since $a \neq b$,

$$DFT(I)(2, 2) = \frac{a - b}{2} \neq 0.$$

Hence

$$\begin{cases} \exp(-\frac{8}{ab}) = \exp(-\frac{1}{4}) & \implies ab = 32 \\ \frac{1}{8}(4a + 4b) = 6 & \implies a + b = 12 \end{cases}$$

Hence $(a, b) = (4, 8)$ or $(8, 4)$.

3. Compute the degradation functions in the frequency domain that correspond to the following $M \times N$ convolution kernels h , i.e. find $H \in M_{M \times N}(\mathbb{C})$ such that

$$DFT(h * f)(u, v) = H(u, v)DFT(f)(u, v)$$

for any periodically extended $f \in M_{M \times N}(\mathbb{R})$:

(a) Assuming integer k satisfies $k \leq \min\{\frac{M}{2}, \frac{N}{2}\}$,

$$h_1(x, y) = \begin{cases} \frac{1}{(2k+1)^2} & \text{if } \text{dist}(x, M\mathbb{Z}) \leq k \text{ and } \text{dist}(y, N\mathbb{Z}) \leq k, \\ 0 & \text{otherwise;} \end{cases}$$

(b) Letting $r > 1$,

$$h_2(x, y) = \begin{cases} \frac{r}{r+4} & \text{if } D(x, y) = 0, \\ \frac{1}{r+4} & \text{if } D(x, y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(c)

$$h_3(x, y) = \begin{cases} \frac{1}{4} & \text{if } D(x, y) = 0, \\ \frac{1}{8} & \text{if } D(x, y) = 1, \\ \frac{1}{16} & \text{if } D(x, y) = 2, \\ 0 & \text{otherwise;} \end{cases}$$

(d)

$$h_4(x, y) = \begin{cases} -4 & \text{if } D(x, y) = 0, \\ 1 & \text{if } D(x, y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(e) Letting $a, b \in \mathbb{Z}$ and $T \in \mathbb{N} \setminus \{0\}$ such that $|a|(T-1) < M$ and $|b|(T-1) < N$,

$$h_5(x, y) = \begin{cases} \frac{1}{T} & \text{if } (x, y) \in \{(at, bt) : t = 0, 1, \dots, T-1\} \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Recall that for any $f \in M_{M \times N}(\mathbb{R})$, $DFT(h * f) = MN \cdot DFT(h) \odot DFT(f)$; hence $H = MN \cdot DFT(h)$.

(a)

$$\begin{aligned} H_1(u, v) &= \sum_{x=-k}^k \sum_{y=-k}^k \frac{1}{(2k+1)^2} e^{-2\pi j(\frac{ux}{M} + \frac{vy}{N})} = \frac{1}{(2k+1)^2} \sum_{x=-k}^k e^{-2\pi j \frac{ux}{M}} \sum_{y=-k}^k e^{-2\pi j \frac{vy}{N}} \\ &= \frac{1}{(2k+1)^2} [1 + 2 \sum_{x=1}^k \cos \frac{2\pi ux}{M}] [1 + 2 \sum_{y=1}^k \cos \frac{2\pi vy}{N}]. \end{aligned}$$

(b)

$$H_2(u, v) = \frac{r}{r+4} + \frac{1}{r+4} (e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}}) = \frac{r + 2(\cos \frac{2\pi u}{M} + \cos \frac{2\pi v}{N})}{r+4}$$

(c)

$$\begin{aligned} H_3(u, v) &= \frac{1}{4} + \frac{1}{8} (e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}}) \\ &\quad + \frac{1}{16} (e^{-2\pi j(\frac{u}{M} + \frac{v}{N})} + e^{-2\pi j(\frac{u}{M} - \frac{v}{N})} + e^{-2\pi j(-\frac{u}{M} + \frac{v}{N})} + e^{-2\pi j(-\frac{u}{M} - \frac{v}{N})}) \\ &= \frac{1}{4} + \frac{1}{4} (\cos \frac{2\pi u}{M} + \cos \frac{2\pi v}{N}) + \frac{1}{4} \cos \frac{2\pi u}{M} \cos \frac{2\pi v}{N} \\ &= \frac{1}{4} (\cos \frac{2\pi u}{M} + 1) (\cos \frac{2\pi v}{N} + 1) \\ &= \cos^2 \frac{\pi u}{M} \cos^2 \frac{\pi v}{N}. \end{aligned}$$

(d)

$$\begin{aligned} H_4(u, v) &= -4 + e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}} \\ &= -4 + 2 \cos \frac{2\pi u}{M} + 2 \cos \frac{2\pi v}{N} \\ &= -4 (\sin^2 \frac{\pi u}{M} + \sin^2 \frac{\pi v}{N}). \end{aligned}$$

(e)

$$\begin{aligned}
H_5(u, v) &= \frac{1}{T} \sum_{t=0}^{T-1} e^{-2\pi j(\frac{atu}{M} + \frac{btv}{N})} \\
&= \begin{cases} \frac{1}{T} \cdot \frac{1-e^{-2\pi jT(\frac{au}{M} + \frac{bv}{N})}}{1-e^{-2\pi j(\frac{au}{M} + \frac{bv}{N})}} & \text{if } \frac{au}{M} + \frac{bv}{N} \notin \mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T} e^{-\pi j(T-1)(\frac{au}{M} + \frac{bv}{N})} \frac{e^{\pi jT(\frac{au}{M} + \frac{bv}{N})} - e^{-\pi jT(\frac{au}{M} + \frac{bv}{N})}}{e^{\pi j(\frac{au}{M} + \frac{bv}{N})} - e^{-\pi j(\frac{au}{M} + \frac{bv}{N})}} & \text{if } \frac{au}{M} + \frac{bv}{N} \notin \mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T} e^{-\pi j(T-1)(\frac{au}{M} + \frac{bv}{N})} \frac{\sin(\pi T(\frac{au}{M} + \frac{bv}{N}))}{\sin(\pi(\frac{au}{M} + \frac{bv}{N}))} & \text{if } \frac{au}{M} + \frac{bv}{N} \notin \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

4. For any periodically extended $N \times N$ image f , define

$$\begin{aligned}
G_x(f)(x, y) &= \frac{1}{4}f(x+1, y) + \frac{1}{2}f(x, y) + \frac{1}{4}f(x-1, y) \\
\text{and } G_y(f)(x, y) &= \frac{1}{4}f(x, y+1) + \frac{1}{2}f(x, y) + \frac{1}{4}f(x, y-1).
\end{aligned}$$

(a) Find an $N \times N$ image h such that for any periodically extended $N \times N$ image f ,

$$h * f = G_x \circ G_y(f).$$

(b) Let $H(u, v)$ be the LPF such that

$$DFT(h * f)(u, v) = H(u, v)DFT(f)(u, v),$$

where h is the convolution kernel from (a). Using H , perform unsharp masking (i.e. $k = 1$) on the following periodically extended 4×4 image

$$f = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Solution:

(a) Note that

$$G_x(f)(x, y) = h_x * f(x, y) \text{ and } G_y(f)(x, y) = h_y * f(x, y),$$

where

$$h_x(x, y) = \begin{cases} \frac{1}{2} & \text{if } (x, y) = (0, 0) \\ \frac{1}{4} & \text{if } (x, y) = (-1, 0) \text{ or } (1, 0) \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_y(x, y) = \begin{cases} \frac{1}{2} & \text{if } (x, y) = (0, 0) \\ \frac{1}{4} & \text{if } (x, y) = (0, -1) \text{ or } (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $G_x(G_y(f)) = h_x * (h_y * f) = (h_x * h_y) * f = h * f$, where

$$\begin{aligned}
h(x, y) &= h_x * h_y(x, y) \\
&= \begin{cases} \frac{1}{4} & \text{if } (x, y) = (0, 0) \\ \frac{1}{8} & \text{if } (x, y) = (0, -1) \text{ or } (-1, 0) \text{ or } (1, 0) \text{ or } (0, 1) \\ \frac{1}{16} & \text{if } (x, y) = (-1, -1) \text{ or } (1, -1) \text{ or } (-1, 1) \text{ or } (1, 1) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) Recall that $DFT(h * f)(u, v) = N^2 DFT(h)(u, v)DFT(f)(u, v)$.

Hence to perform unsharp masking on $f \in M_{4 \times 4}$,

$$\begin{aligned}
H &= 16DFT(h) \\
&= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 4 & 2 & 0 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 8 & 4 & 0 & 4 \\ 4 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 16 & 8 & 0 & 8 \\ 8 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 8 & 4 & 0 & 4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 & 2 & 0 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
DFT(f) &= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 6 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 8 & 6 & 4 & 6 \\ 6 & 4 & 2 & 4 \\ 4 & 2 & 0 & 2 \\ 6 & 4 & 2 & 4 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 4 & 3 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{pmatrix}.
\end{aligned}$$

$\tilde{F}(u, v) = DFT(f)(u, v)[2 - H(u, v)]$ and thus

$$\begin{aligned}
\tilde{F} &= \frac{1}{32} \begin{pmatrix} 4 & 3 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{pmatrix} \odot \begin{pmatrix} 4 & 6 & 8 & 6 \\ 6 & 7 & 8 & 7 \\ 8 & 8 & 8 & 8 \\ 6 & 7 & 8 & 7 \end{pmatrix} \\
&= \frac{1}{32} \begin{pmatrix} 16 & 18 & 16 & 18 \\ 18 & 14 & 8 & 14 \\ 16 & 8 & 0 & 8 \\ 18 & 14 & 8 & 14 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 8 & 9 & 8 & 7 \\ 9 & 7 & 4 & 7 \\ 8 & 4 & 0 & 4 \\ 9 & 7 & 4 & 7 \end{pmatrix} \\
\text{and } \tilde{f} &= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} 8 & 9 & 8 & 7 \\ 9 & 7 & 4 & 7 \\ 8 & 4 & 0 & 4 \\ 9 & 7 & 4 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 34 & 27 & 16 & 27 \\ 0 & 5 & 8 & 5 \\ -2 & -1 & 0 & -1 \\ 0 & 5 & 8 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} 104 & 18 & -4 & 18 \\ 18 & -8 & -2 & -8 \\ -4 & -2 & 0 & -2 \\ 18 & -8 & -2 & -8 \end{pmatrix} \\
&= \frac{1}{8} \begin{pmatrix} 52 & 9 & -2 & 9 \\ 9 & -4 & -1 & -4 \\ -2 & -1 & 0 & -1 \\ 9 & -4 & -1 & -4 \end{pmatrix}.
\end{aligned}$$

5. Let $W_N(n, k) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$ for $0 \leq n, k \leq N-1$ and $W = W_N \otimes W_N$. Suppose $N = 4$, we have following problems:

(a) Prove that $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

(b) Show that $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N \times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

Solution: Method 1 (for general N):

(a) Recall that the definition of Kronecker product is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

We have

$$W = W_N \otimes W_N = \begin{pmatrix} \frac{1}{\sqrt{N}}e^{2\pi j \frac{0 \cdot 0}{N}} \cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j \frac{0 \cdot 1}{N}} \cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j \frac{0 \cdot (N-1)}{N}} \cdot W_N \\ \frac{1}{\sqrt{N}}e^{2\pi j \frac{1 \cdot 0}{N}} \cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j \frac{1 \cdot 1}{N}} \cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j \frac{1 \cdot (N-1)}{N}} \cdot W_N \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{N}}e^{2\pi j \frac{(N-1) \cdot 0}{N}} \cdot W_N & \frac{1}{\sqrt{N}}e^{2\pi j \frac{(N-1) \cdot 1}{N}} \cdot W_N & \cdots & \frac{1}{\sqrt{N}}e^{2\pi j \frac{(N-1) \cdot (N-1)}{N}} \cdot W_N \end{pmatrix}$$

And rewrite it as $W = \left(\frac{1}{\sqrt{N}}e^{2\pi j \frac{n \cdot k}{N}} \cdot W_N \right)_{0 \leq n, k \leq N-1}$

Since $\overline{W_N} = \left(\frac{1}{\sqrt{N}}e^{-2\pi j \frac{n \cdot k}{N}} \right)_{0 \leq n, k \leq N-1}$, we know that

$$\overline{W_N} \otimes \overline{W_N} = \begin{pmatrix} \frac{1}{\sqrt{N}}e^{-2\pi j \frac{0 \cdot 0}{N}} \cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j \frac{0 \cdot 1}{N}} \cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j \frac{0 \cdot (N-1)}{N}} \cdot \overline{W_N} \\ \frac{1}{\sqrt{N}}e^{-2\pi j \frac{1 \cdot 0}{N}} \cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j \frac{1 \cdot 1}{N}} \cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j \frac{1 \cdot (N-1)}{N}} \cdot \overline{W_N} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{N}}e^{-2\pi j \frac{(N-1) \cdot 0}{N}} \cdot \overline{W_N} & \frac{1}{\sqrt{N}}e^{-2\pi j \frac{(N-1) \cdot 1}{N}} \cdot \overline{W_N} & \cdots & \frac{1}{\sqrt{N}}e^{-2\pi j \frac{(N-1) \cdot (N-1)}{N}} \cdot \overline{W_N} \end{pmatrix}$$

And rewrite it as $\overline{W_N} \otimes \overline{W_N} = \left(\frac{1}{\sqrt{N}}e^{-2\pi j \frac{n \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \leq n, k \leq N-1}$

Using block-matrix multiplication, we can calculate

$$\begin{aligned} W \cdot (\overline{W_N} \otimes \overline{W_N}) &= \left(\frac{1}{\sqrt{N}}e^{2\pi j \frac{n \cdot k}{N}} \cdot W_N \right)_{0 \leq n, k \leq N-1} \left(\frac{1}{\sqrt{N}}e^{-2\pi j \frac{n \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \leq n, k \leq N-1} \\ &= \left(\sum_{p=0}^{N-1} \frac{1}{\sqrt{N}}e^{2\pi j \frac{n \cdot p}{N}} \cdot W_N \cdot \frac{1}{\sqrt{N}}e^{-2\pi j \frac{p \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \leq n, k \leq N-1} \\ &= \left(\sum_{p=0}^{N-1} \frac{1}{N}e^{2\pi j \frac{(n-k) \cdot p}{N}} \cdot W_N \overline{W_N} \right)_{0 \leq n, k \leq N-1} \\ &= I_{N^2} \end{aligned}$$

Therefore, $W^{-1} = \overline{W_N} \otimes \overline{W_N}$.

(b) Note that $f = (f_{i,j})_{0 \leq i, j \leq N-1} \in M_{N \times N}(\mathbb{C})$ then

$$\mathcal{S}(f) = (f_{0,0} \quad f_{1,0} \quad \cdots \quad f_{N-1,0} \quad \cdots \quad f_{0,N-1} \quad f_{1,N-1} \quad \cdots \quad f_{N-1,N-1})^T \in M_{N^2 \times 1}(\mathbb{C})$$

From (a) we know that $W^{-1} = \overline{W_N} \otimes \overline{W_N} = \left(\frac{1}{\sqrt{N}}e^{-2\pi j \frac{n \cdot k}{N}} \cdot \overline{W_N} \right)_{0 \leq n, k \leq N-1} \in M_{N^2 \times N^2}(\mathbb{C})$

Then, $W^{-1}\mathcal{S}(f) \in M_{N^2 \times 1}(\mathbb{C})$, and its l -th entry is

$$\begin{aligned} (W^{-1}\mathcal{S}(f))_l &= W^{-1}(l, :) \cdot \mathcal{S}(f) \\ &= \begin{pmatrix} \frac{1}{\sqrt{N}} e^{-2\pi j \lfloor \frac{l}{N} \rfloor \cdot 0} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\text{mod}_N(l) \cdot 0}{N}} \\ \frac{1}{\sqrt{N}} e^{-2\pi j \lfloor \frac{l}{N} \rfloor \cdot 0} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\text{mod}_N(l) \cdot 1}{N}} \\ \vdots \\ \frac{1}{\sqrt{N}} e^{-2\pi j \lfloor \frac{l}{N} \rfloor \cdot 0} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\text{mod}_N(l) \cdot (N-1)}{N}} \\ \vdots \\ \frac{1}{\sqrt{N}} e^{-2\pi j \lfloor \frac{l}{N} \rfloor \cdot (N-1)} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\text{mod}_N(l) \cdot 0}{N}} \\ \frac{1}{\sqrt{N}} e^{-2\pi j \lfloor \frac{l}{N} \rfloor \cdot (N-1)} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\text{mod}_N(l) \cdot 1}{N}} \\ \vdots \\ \frac{1}{\sqrt{N}} e^{-2\pi j \lfloor \frac{l}{N} \rfloor \cdot (N-1)} \cdot \frac{1}{\sqrt{N}} e^{-2\pi j \frac{\text{mod}_N(l) \cdot (N-1)}{N}} \end{pmatrix}^T \begin{pmatrix} f_{0,0} \\ f_{1,0} \\ \vdots \\ f_{N-1,0} \\ \vdots \\ f_{0,N-1} \\ f_{1,N-1} \\ \vdots \\ f_{N-1,N-1} \end{pmatrix} \\ &= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{1}{N} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot q + \text{mod}_N(l) \cdot p}{N}} f_{p,q} \end{aligned}$$

$$\text{From } \hat{f}_{p,q} = \frac{1}{N^2} \sum_{\alpha,\beta=0}^{N-1} f_{\alpha,\beta} \cdot e^{-2\pi j \frac{p\alpha+q\beta}{N}}, \text{ we have } \mathcal{S}(\hat{f}) = \begin{pmatrix} \hat{f}_{0,0} \\ \hat{f}_{1,0} \\ \vdots \\ \hat{f}_{N-1,0} \\ \vdots \\ \hat{f}_{0,N-1} \\ \hat{f}_{1,N-1} \\ \vdots \\ \hat{f}_{N-1,N-1} \end{pmatrix} \in M_{N^2 \times 1}(\mathbb{C})$$

$$\text{and } (N \cdot (\mathcal{S}(f)))_l = N \hat{f}(\text{mod}_N(l), \lfloor \frac{l}{N} \rfloor) = N \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{1}{N^2} e^{-2\pi j \frac{\lfloor \frac{l}{N} \rfloor \cdot q + \text{mod}_N(l) \cdot p}{N}} f_{p,q}$$

Therefore, $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$ for any $f \in M_{N \times N}(\mathbb{C})$, where $\hat{f} = DFT(f)$.

Method 2 (directly compute when $N = 4$):

(a) We can directly calculate that

$$W_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}$$

and

$$\overline{W}_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}.$$

It can be checked that $W_4 \overline{W}_4 = I_4$, which means $\overline{W}_4 = W_4^{-1}$. Therefore

$$W^{-1} = W_4^{-1} \otimes W_4^{-1} = \overline{W}_4 \otimes \overline{W}_4.$$

(b) By the matrix form of DFT, we have that

$$\hat{f} = DFT(f) = \frac{1}{16} (2\overline{W}_4) f (2\overline{W}_4) = \frac{1}{4} \overline{W}_4 f \overline{W}_4.$$

So it can be treated as a spearable linear transformation with transformation matrix

$$\overline{W}_4^T \otimes (\frac{1}{4} \overline{W}_4) = \frac{1}{4} W^{-1}.$$

Then

$$W^{-1}\mathcal{S}(f) = 4 \left(\frac{1}{4} W^{-1} \mathcal{S}(f) \right) = 4\mathcal{S}(\hat{f}).$$

6. Let $f = (f_{ij})_{0 \leq i, j \leq N-1} \in M_{N \times N}(\mathbb{R})$ be a clean image. Suppose f is blurred to g under a motion which is given by:

$$g(x, y) = \sum_{t=0}^3 f(x+t, y).$$

Show that $DFT(g)(u, v) = H(u, v)DFT(f)(u, v)$ and find $H(u, v)$.

Solution:

$$\begin{aligned} DFT(g)(u, v) &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} g(x, y) e^{-2\pi j \frac{ux+vy}{N}} \\ &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \sum_{t=0}^3 f(x+t, y) e^{-2\pi j \frac{ux+vy}{N}} \\ &= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \sum_{t=0}^3 f(x+t, y) e^{-2\pi j \frac{u(x+t)+vy}{N}} e^{-2\pi j \frac{-ut}{N}} \\ &= \frac{1}{N^2} \sum_{t=0}^3 \sum_{x'=t}^{N-1+t} \sum_{y=0}^{N-1} f(x', y) e^{-2\pi j \frac{ux'+vy}{N}} e^{-2\pi j \frac{-ut}{N}} \\ &= \left(\sum_{t=0}^3 e^{2\pi j \frac{ut}{N}} \right) \left[\frac{1}{N^2} \sum_{x'=0}^{N-1} \sum_{y=0}^{N-1} f(x', y) e^{-2\pi j \frac{ux'+vy}{N}} \right] \\ &= \begin{cases} \frac{1 - e^{8\pi j \frac{u}{N}}}{1 - e^{2\pi j \frac{u}{N}}} DFT(f)(u, v) & \text{if } u \neq 0, \\ 4 DFT(f)(u, v) & \text{otherwise,} \end{cases} \end{aligned}$$

Hence $H(u, v) = \begin{cases} \frac{1 - e^{8\pi j \frac{u}{N}}}{1 - e^{2\pi j \frac{u}{N}}} & \text{if } u \neq 0, \\ 4 & \text{otherwise,} \end{cases}$

7. Given $N^2 \times N^2$ block-circulant real matrices D and L , $N \times N$ image g and fixed parameter $\varepsilon > 0$, the constrained least square filtering aims to find $f \in M_{N \times N}$ that minimizes:

$$E(f) = [LS(f)]^T [LS(f)]$$

subject to the constraint:

$$[S(g) - DS(f)]^T [S(g) - DS(f)] = \varepsilon,$$

where S is the stacking operator. Let $W = W_N \otimes W_N$, where $W_N(n, k) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$, we have D and L is diagonalizable by W , i.e. $\Lambda_D = W^{-1}DW$ and $\Lambda_L = W^{-1}LW$ is diagonal. Given that the optimal solution f that solves the constrained least square problem satisfies

$$[\lambda D^T D + L^T L]S(f) = \lambda D^T S(g)$$

for some parameter λ . Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$ and λ , where $LS(f) = S(p * f)$ and $DS(f) = S(h * f)$ for any $f \in M_{N \times N}(\mathbb{R})$.

Solution:

Let $\vec{f} = S(f)$ and $\vec{g} = S(g)$. It's easy to know $D = W\Lambda_D W^{-1}$ and $L = W\Lambda_L W^{-1}$. Hence

$$\begin{aligned} (\lambda D^T D + L^T L)\vec{f} &= (\lambda D^* D + L^* L)\vec{f} \\ &= \{\lambda [W\Lambda_D W^{-1}]^* W\Lambda_D W^{-1} + [W\Lambda_L W^{-1}]^* W\Lambda_L W^{-1}\}\vec{f} \\ &= [\lambda W\Lambda_D^* \Lambda_D W^{-1} + W\Lambda_L^* \Lambda_L W^{-1}]\vec{f} \\ &= W(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L)W^{-1}\vec{f} \\ &= W(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L)W^{-1}NS(DFT(f)); \end{aligned}$$

on the other hand,

$$\begin{aligned} \lambda D^T \vec{g} &= \lambda D^* \vec{g} \\ &= \lambda W\Lambda_D^* W^{-1}\vec{g} \\ &= \lambda W\Lambda_D^* N S(DFT(g)). \end{aligned}$$

Hence $(\lambda \Lambda_D^* \Lambda_D + \Lambda_L^* \Lambda_L) \mathcal{S}(DFT(f)) = \lambda \Lambda_D^* \mathcal{S}(DFT(g))$.

From Theorem 1 in lecture 16, we know that for any $k, l, x, y \in \{0, 1, \dots, N-1\}$

$$\Lambda_D(x + kN, y + lN) = \begin{cases} N^2 DFT(h)(x, k) & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda_L(x + kN, y + lN) = \begin{cases} N^2 DFT(p)(x, k) & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise} \end{cases}$$

By comparing each pair of entries, we have

$$(\lambda N^4 |DFT(h)(u, v)|^2 + N^4 |DFT(p)(u, v)|^2) DFT(f)(u, v) = \lambda N^2 DFT(h)(u, v) DFT(g)(u, v),$$

which yields

$$DFT(f)(u, v) = \frac{\lambda DFT(h)(u, v) DFT(g)(u, v)}{N^2 (\lambda |DFT(h)(u, v)|^2 + |DFT(p)(u, v)|^2)}.$$

8. Given $g \in M_{N \times N}(\mathbb{R})$, block circulant $D, L_1, L_2 \in M_{N^2 \times N^2}(\mathbb{R})$ and $\varepsilon > 0$, we aim to minimize $\|L_1 \vec{f}\|_2^2 + \|L_2 \vec{f}\|_2^2$ subject to $\|\vec{g} - D\vec{f}\|_2^2 = \varepsilon$ over $f \in M_{N \times N}(\mathbb{R})$, where $\vec{f} = \mathcal{S}(f)$ and $\vec{g} = \mathcal{S}(g)$ vectorized by the stack operator $\mathcal{S}$.

Given Lagrange multiplier λ for the equality constraint, show that if f is a minimizer of the above constrained minimization problem, then

$$(\lambda D^T D + L_1^T L_1 + L_2^T L_2) \vec{f} = \lambda D^T \vec{g}.$$

Please prove your answer with details.

Solution: Please check the solution of hw 5.

9. Given a 2D simple connected domain D and a noisy image $I : D \rightarrow \mathbb{R}$. Assume $I = 0$ on the boundary of D . We consider the following image denoising model to restore the original clean image $f : D \rightarrow \mathbb{R}$ that minimizes:

$$E(f) = \int_D (f(x, y) - I(x, y))^2 dx dy + \int_D \sqrt{|\nabla f(x, y)|^2 + \epsilon} dx dy$$

where small parameter $\epsilon > 0$.

- (a) If f minimizes $E(f)$, show that f should satisfy the following conditions:

$$2f(x, y) - 2I(x, y) - \nabla \cdot \left(\frac{\nabla f(x, y)}{\sqrt{|\nabla f(x, y)|^2 + \epsilon}} \right) = 0 \text{ for } (x, y) \in D.$$

- (b) Derive an iterative scheme, which updates f_n to f_{n+1} with time step $\tau > 0$, to minimize $E(f)$.

Solution:

- (a) Suppose f minimizes $E(f)$, then for any $v : D \rightarrow \mathbb{R}$,

$$\begin{aligned} 0 &= \frac{d}{dt} E(f + tv) \Big|_{t=0} \\ &= \int_D \frac{d}{dt} \Big|_{t=0} \left[(f + tv - I)^2 + \sqrt{|\nabla(f + tv)|^2 + \epsilon} \right] dx dy \\ &= \int_D \left[2(f - I)v + 2tv^2 + \frac{(\nabla f + t\nabla v) \cdot \nabla v}{\sqrt{|\nabla f + t\nabla v|^2 + \epsilon}} \right] \Big|_{t=0} dx dy \\ &= \int_D \left[2(f - I)v + \frac{\nabla f \cdot \nabla v}{\sqrt{|\nabla f|^2 + \epsilon}} \right] dx dy \\ &= \int_D \left[2(f - I) - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \right] v dx dy + \int_{\partial D} \left\langle \frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}}, \vec{n} \right\rangle v ds \end{aligned}$$

We have $I = 0$ on ∂D , so $\vec{n} = 0$ holds on ∂D and the last term can be removed as

$$0 = \int_D \left[2(f - I) - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \right] v dx dy.$$

Since the above equation holds for any v , it could be

$$2f(x, y) - 2I(x, y) - \nabla \cdot \left(\frac{\nabla f(x, y)}{\sqrt{|\nabla f(x, y)|^2 + \epsilon}} \right) = 0 \text{ for } (x, y) \in D.$$

(b) Based on the derivation above, dropping the assumption that f is a minimizer of E , we have:

$$\begin{aligned} \frac{d}{dt} E(f + tv) &= \int_D \frac{d}{dt} \left[(f + tv - I)^2 + \sqrt{|\nabla(f + tv)|^2 + \epsilon} \right] dx dy \\ &= \int_D 2(f - I)v + 2tv^2 + \frac{(\nabla f + t\nabla v) \cdot \nabla v}{\sqrt{|\nabla f + t\nabla v|^2 + \epsilon}} dx dy \\ &\approx \int_D 2(f - I)v + \frac{\nabla f \cdot \nabla v}{\sqrt{|\nabla f|^2 + \epsilon}} dx dy \quad \text{for small } t \\ &= \int_D \left[2(f - I) - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \right] v dx dy \end{aligned}$$

For v to be a descent direction, we need $\frac{d}{dt} E(f + tv) < 0$.

So choosing $v = - \left[2(f - I) - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \right]$ on D , the integrals will be both less than 0. Hence a descent direction is:

$$-2f(x, y) + 2I(x, y) + \nabla \cdot \left(\frac{\nabla f(x, y)}{\sqrt{|\nabla f(x, y)|^2 + \epsilon}} \right) \text{ for } (x, y) \in D,$$

and thus $E(f)$ can be iteratively minimized by updating f :

$$f^{n+1}(x, y) = f^n(x, y) - \tau \left[2f(x, y) - 2I(x, y) - \nabla \cdot \left(\frac{\nabla f(x, y)}{\sqrt{|\nabla f(x, y)|^2 + \epsilon}} \right) \right] \text{ if } (x, y) \in D$$

for a small time step $\tau > 0$.

10. Given a noisy image $I : D \rightarrow \mathbb{R}$, we consider the following image denoising model to restore the original clean image $f : D \rightarrow \mathbb{R}$ that minimizes:

$$E(f) = \int_D (f(x, y) - I(x, y))^2 dx dy + \int_D |\nabla f(x, y)|^2 dx dy.$$

where $|\nabla f(x, y)|^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$. Assume $I = 0$ on the boundary of D .

- (a) Derive an iterative scheme, which updates f^n to f^{n+1} with time step $\tau > 0$, to minimize $E(f)$. This is the same model as discussed in the lectures. Please show all your steps with details, including detailed explanations on why E is iteratively decreasing. Missing detailed steps will result in mark deductions.
- (b) If E is modified to $\tilde{E}$ defined as follows:

$$\tilde{E}(f) = \int_D \sqrt{(f(x, y) - I(x, y))^2 + \epsilon^2} dx dy + \int_D \sqrt{|\nabla f(x, y)|^2 + \epsilon^2} dx dy,$$

where $\epsilon > 0$ is a small parameter bigger than 0. Derive an iterative scheme, which updates f^n to f^{n+1} with time step $\tau > 0$, to minimize $\tilde{E}(f)$. Please show all your steps with details, including detailed explanations on why $\tilde{E}$ is iteratively decreasing. Missing detailed steps will result in mark deductions.

Solution:

(a) Suppose f minimizes $E(f)$, then for any $v : D \rightarrow \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{d}{dt} E(f + tv) \Big|_{t=0} \\
&= \int_D \frac{d}{dt} \Big|_{t=0} [(f + tv - I)^2 + \langle \nabla(f + tv), \nabla(f + tv) \rangle] dx dy \\
&= \int_D 2 [(f - I)v + tv^2 + \langle \nabla f, \nabla v \rangle + t \langle \nabla v, \nabla v \rangle] \Big|_{t=0} dx dy \\
&= \int_D 2 [(f - I)v + \nabla f \cdot \nabla v] dx dy \\
&= \int_D 2 [(f - I) - \nabla \cdot \nabla f] v dx dy + \int_{\partial D} 2 \langle \nabla f, \vec{n} \rangle v ds
\end{aligned}$$

We have $I = 0$ on ∂D , so $\vec{n} = 0$ there, the equation becomes

$$\int_D 2 [(f - I) - \Delta f] v dx dy = 0.$$

Since the above equation holds for any v , it must be

$$f(x, y) - I(x, y) - \Delta f(x, y) = 0 \text{ for } (x, y) \in D$$

Refer to the argument in 9(b), a descent direction is:

$$-2f(x, y) + 2I(x, y) + 2\Delta f(x, y) \text{ for } (x, y) \in D,$$

and thus $E(f)$ can be iteratively minimized by updating f :

$$f^{n+1}(x, y) = f^n(x, y) - 2\tau [f(x, y) - I(x, y) - \Delta f(x, y)] \text{ if } (x, y) \in D$$

for a small time step $\tau > 0$.

(b) Suppose f minimizes $\tilde{E}(f)$, then for any $v : D \rightarrow \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{d}{dt} \tilde{E}(f + tv) \Big|_{t=0} \\
&= \int_D \frac{d}{dt} \Big|_{t=0} \left[\sqrt{(f + tv - I)^2 + \epsilon^2} + \sqrt{|\nabla(f + tv)|^2 + \epsilon^2} \right] dx dy \\
&= \int_D \left[\frac{(f - I)v + tv^2}{\sqrt{(f + tv - I)^2 + \epsilon^2}} + \frac{(\nabla f + t \nabla v) \cdot \nabla v}{\sqrt{|\nabla f + t \nabla v|^2 + \epsilon^2}} \right] \Big|_{t=0} dx dy \\
&= \int_D \left[\frac{(f - I)v}{\sqrt{(f - I)^2 + \epsilon^2}} + \frac{\nabla f \cdot \nabla v}{\sqrt{|\nabla f|^2 + \epsilon}} \right] dx dy \\
&= \int_D \left[\frac{f - I}{\sqrt{(f - I)^2 + \epsilon^2}} - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \right] v dx dy
\end{aligned}$$

Since the above equation holds for any v , it must be

$$\frac{f - I}{\sqrt{(f - I)^2 + \epsilon^2}} - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) = 0.$$

Refer to the argument in 9(b), a descent direction is:

$$-\frac{f - I}{\sqrt{(f - I)^2 + \epsilon^2}} + \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \text{ for } (x, y) \in D,$$

and thus $E(f)$ can be iteratively minimized by updating f :

$$f^{n+1}(x, y) = f^n(x, y) - \tau \left[\frac{f - I}{\sqrt{(f - I)^2 + \epsilon^2}} - \nabla \cdot \left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + \epsilon}} \right) \right] \text{ if } (x, y) \in D$$

for a small time step $\tau > 0$.