# MMAT5390: Mathematical Image Processing

## Assignment 5

Due: 23:59 Tuesday, April 22, 2025

Please give reasons in your solutions unless otherwise specified.

1. Given three $N^2 \times N^2$ block-circulant real matrices $D$, $L_1$, $L_2$, and a $N \times N$ image $g$, we aim to find an image $f \in M_{N\times N}$ that minimizes:

$$\|L_1\vec{f}\|_2^2 + \|L_2\vec{f}\|_2^2$$

subject to the constraint:

$$\|\vec{g} - D\vec{f}\|_2^2 = \varepsilon,$$

where $\vec{f} = \mathcal{S}(f)$ and $\vec{g} = \mathcal{S}(g)$ vectorized by the stack operator $\mathcal{S}$, and $\varepsilon > 0$ is a fixed parameter.

   (a) Given Lagrange multiplier $\lambda$ for the equality constraint, show that the optimal solution $f$ that solves the above constrained minimization problem satisfies

$$(\lambda D^T D + L_1^T L_1 + L_2^T L_2)\vec{f} = \lambda D^T \vec{g}.$$

   (b) Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$, $DFT(q)$ and $\lambda$, where $D\mathcal{S}(\varphi) = \mathcal{S}(h * \varphi)$, $L_1\mathcal{S}(\varphi) = \mathcal{S}(p * \varphi)$, and $L_2\mathcal{S}(\varphi) = \mathcal{S}(q * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$. (**hints**: You may use $W^{-1}\mathcal{S}(f) = N\mathcal{S}(\hat{f})$, where $\hat{f} = DFT(f)$.)

2. Suppose $I = (I(k,l))_{0\le k,l\le 3}$ is a periodically extended $4\times 4$ image given by:

$$I = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Find the image after applying the following filters respectively.

   (a) $3\times 3$ mean filter;

   (b) $3\times 3$ median filter;

   (c) convolution filter $K_{0\le k,l\le 3} = \frac{1}{4}\begin{pmatrix} 4 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

3. Consider a periodically extended $4\times 4$ image $I = (I(x,y))_{0\le x,y\le 3}$ given by:

$$I = \begin{pmatrix} 1 & a & 0 & 0 \\ b & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & 1 \end{pmatrix}$$

Given that the discrete Laplacian $\Delta I$ of $I$ is given by the formula:

$$\Delta I(x,y) = -4I(x,y) + I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) \text{ for } 0 \le x,y \le 3.$$

We perform the Laplacian masking on $I$ to get a sharpen image $I_{sharp}$, where $I_{sharp} = I - \Delta I$. Suppose $I_{sharp}$ is given by

$$I_{sharp} = \begin{pmatrix} 4 & -6 & 1 & -3 \\ 8 & -1 & -2 & 3 \\ -2 & -1 & 5 & -3 \\ -2 & 1 & -2 & 5 \end{pmatrix}.$$

Find $a$, $b$ and $c$.

4. Consider the following energy functional for image denoising

$$E(f) = \int_\Omega \left[(f(x,y) - g(x,y))^2 + \lambda \|\nabla f(x,y)\|^2\right] \, dx \, dy.$$

where:

- $\Omega \subset \mathbb{R}^2$ is the image domain,
- $g : \Omega \to \mathbb{R}$ is the observed noisy image,
- $f : \Omega \to \mathbb{R}$ is the unknown clean image to be recovered,
- $\lambda > 0$ is a regularization parameter.

(a) Derive the Euler-Lagrange equation for $E(f)$ in the continuous setting. And find a necessary condition for optimality.

(b) Show whether the derived Euler-Lagrange equation guarantees a unique minimizer, i.e. verify whether the necessary condition is sufficient to guarantee optimality.

(c) Propose a gradient descent-based iterative method to minimize $E(f)$.

5. **(Coding assignment, optional)**
Instruction: Please read the MATLAB script in the attached zip file carefully. There are several lines missing in this script. Add the missing lines by yourself and test it using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)
The inverse Haar transform can be written as follows

$$I(m,n) = \sum_{i=1}^{N} \tilde{H}^T(m,i) \sum_{j=1}^{N} \hat{I}(i,j)\tilde{H}(j,n) \tag{1}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{I}(i,j)\tilde{H}^T(m,i)\tilde{H}(j,n) \tag{2}$$

where $I$ and $\hat{I}$ are the reconstructed image and the Haar coefficients respectively. From the above equations, we know that the transform can be rewritten as the summation of weighted elementary images. Now, we will implement this reconstruction process.