Homework 3 Solutions 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
Prepared by Tianxu Lan
Please send corrections, if any, to 1155184513@link.cuhk.edu.hk

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1.

(a) Let $X : \Omega \to \mathbb{R}$ be a random variable such that $X \equiv 0$, i.e. for any $\omega \in \Omega$ $X(\omega) = 0$., Prove that $\sigma(X) = \{\emptyset, \Omega\}$.
(b) Let $G := \{\emptyset, \Omega\}$, and $X : \Omega \to \mathbb{R}$ be G-measurable. Prove that $X \equiv c$ for some constant $c \in \mathbb{R}$.

Solution. (a) Let $x \in \mathbb{R}$. We have

$$\{\omega \in \Omega : X(\omega) \leq x\} = \begin{cases} \Omega, & x \geq 0, \\ \emptyset, & x < 0. \end{cases}$$

Thus, $\{\emptyset, \Omega\} \subseteq \sigma(X)$. The other direction follows from the definition of a sigma algebra.
(b) There is more than one way to prove this proposition; here we just provide one particular proof. Since X is G-measurable, it follows that $\sigma(X) \subseteq G = \{\emptyset, \Omega\}$. Pick an arbitrary $\omega_0 \in \Omega$ and for this ω_0, we have that

$$\{\omega \in \Omega : X(\omega) = X(\omega_0)\} \in \sigma(X) \subseteq \{\emptyset, \Omega\}.$$

Hence, either $\{\omega \in \Omega : X(\omega) = X(\omega_0)\} = \emptyset$ or $\{\omega \in \Omega : X(\omega) = X(\omega_0)\} = \Omega$. The former case is impossible by our construction and so the latter case yields the result for $c := X(\omega_0)$.

2.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$ be a filtration. Given an $\mathcal{F}$-predictable process $(H_n)_{n \geq 0}$, which is uniformly bounded, and an $\mathcal{F}$-martingale $(X_n)_{n \geq 0}$, we define a process $(V_n)_{n \geq 0}$ by

$$V_0 := 0, \quad V_n := \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

Prove that $(V_n)_{n\geq 0}$ is still an $\mathcal{F}$-martingale.

Solution. V is $\mathcal{F}$-adapted because each summand $H_k(X_k - X_{k-1})$ is $\mathcal{F}_n$-measurable for $1 \leq k \leq n$.
Each V_n is integrable because each summand is integrable. Indeed, let C denote the uniform bound for H_k, we have that

$$E[|H_k(X_k - X_{k-1})|] \leq C \cdot (E[|X_k|] + E[|X_{k-1}|]) < \infty.$$

For the martingale property, it follows from linearity of conditional expectation that

$$E[V_{n+1}|\mathcal{F}_n] = \sum_{k=1}^{n+1} E[H_k(X_k - X_{k-1})|\mathcal{F}_n]$$

$$= \sum_{k=1}^{n} E[H_k(X_k - X_{k-1})|\mathcal{F}_n] + E[H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n]$$

$$= \sum_{k=1}^{n} H_k(X_k - X_{k-1}) + 0 = V_n.$$

where we have used that $H_k(X_k - X_{k-1})$ is $\mathcal{F}_n$-measurable for $1 \leq k \leq n$ and

$$E[H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n] = H_{n+1}E[(X_{n+1} - X_n)|\mathcal{F}_n]$$

$$= H_{n+1}(E[X_{n+1}|\mathcal{F}_n] - E[X_n|\mathcal{F}_n]) = H_{n+1}(X_n - X_n) = 0.$$

3.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be a filtration. Given an $\mathcal{F}$-submartingale $(X_n)_{n\geq 0}$, we define

$$\Delta A_n := E[X_n|\mathcal{F}_{n-1}] - X_{n-1}, \quad \Delta M_n := X_n - E[X_n|\mathcal{F}_{n-1}], \quad \forall n \geq 1,$$

and

$$A_0 = M_0 = 0, \quad A_n := \sum_{k=1}^{n} \Delta A_k, \quad M_n := \sum_{k=1}^{n} \Delta M_k.$$

(a) Prove that $(M_n)_{n\geq 0}$ is an $\mathcal{F}$-martingale, and that $(A_n)_{n\geq 0}$ is an increasing $\mathcal{F}$-predictable process.

(b) Prove that $(X_n)_{n\geq 0}$ has the decomposition

$$X_n = X_0 + M_n + A_n, \quad \forall n \geq 0.$$

(c) Let $(A_1^n)_{n\geq 0}$ and $(A_2^n)_{n\geq 0}$ be two $\mathcal{F}$-predictable processes such that $A_1^0 = A_2^0 = 0$. Prove that if $(A_1^n - A_2^n)_{n\geq 0}$ is an $\mathcal{F}$-martingale, then $A_1^n = A_2^n$, a.s. for each $n \geq 1$.

(d) Deduce that the decomposition (1) is unique, i.e. if one has

$$X_n = X_0 + \tilde{M}_n + \tilde{A}_n, \quad \forall n \geq 0,$$

for some $\mathcal{F}$-martingale $(\tilde{M}_n)_{n\geq 0}$ and increasing $\mathcal{F}$-predictable process $(\tilde{A}_n)_{n\geq 0}$ such that $\tilde{M}_0 = \tilde{A}_0 = 0$, then $A_n = \tilde{A}_n$ and $M_n = \tilde{M}_n$, a.s. for each $n \geq 1$.

Proof. (a) We will omit the arguments that M is $\mathcal{F}$-adapted and that each M_n is integrable. We only show the martingale property, which is the only non-trivial part. The same goes for the rest of this problem.

$$E[M_{n+1}|\mathcal{F}_n] = \sum_{k=1}^{n+1} E[\Delta M_k|\mathcal{F}_n]$$

$$= E[\Delta M_{n+1}|\mathcal{F}_n] + \sum_{k=1}^{n} E[\Delta M_k|\mathcal{F}_n] = 0 + M_n = M_n,$$

where we have used

$$E[\Delta M_{n+1}|\mathcal{F}_n] = E[X_{n+1}|\mathcal{F}_n] - E[E[X_{n+1}|\mathcal{F}_n]|\mathcal{F}_n] = 0.$$

While for A_n, predictability follows from

$$\begin{aligned}
&E[X_k|\mathcal{F}_{k-1}] \in \mathcal{F}_{k-1} \subseteq \mathcal{F}_{n-1} \quad \text{and} \quad X_{k-1} \in \mathcal{F}_{k-1} \subseteq \mathcal{F}_{n-1} \\
\Rightarrow &\Delta A_k = E[X_k|\mathcal{F}_{k-1}] - X_{k-1} \in \mathcal{F}_{n-1}. \\
\Rightarrow &A_n = \sum_{k=1}^{n} \Delta A_k \in \mathcal{F}_{n-1}
\end{aligned}$$

and it is increasing because

$$A_{n+1} - A_n = \Delta A_{n+1} = E[X_{n+1}|\mathcal{F}_n] - X_n \geq X_n - X_n = 0.$$

(b)

$$X_0 + M_n + A_n = X_0 + \sum_{k=1}^{n} \Delta M_k + \Delta A_k = X_0 + \sum_{k=1}^{n} (X_k - X_{k-1}) = X_0 + (X_n - X_0) = X_n.$$

(c) Denote $B_n := A_1^n - A_2^n$. On one hand, since it is predictable, we have

$$E[B_n|\mathcal{F}_{n-1}] = B_n \text{ a.s.}$$

On the other hand, since it is a martingale, we have

$$E[B_n|\mathcal{F}_{n-1}] = B_{n-1} \text{ a.s.}$$

Hence, we have $B_n = B_{n-1}$ a.s. and the result follows as an immediate consequence.

(d) By the hypothesis,

$$M_n + A_n = \tilde{M}_n + \tilde{A}_n.$$

Rearranging gives

$$M_n - \tilde{M}_n = \tilde{A}_n - A_n.$$

Since both M_n and $\tilde{M}_n$ are martingales, it follows that $\tilde{A}_n - A_n$ is a martingale. Hence by part (c), $\tilde{A}_n = A_n$ a.e. and so $\tilde{M}_n = M_n$ a.s. as well.