Homework 1 Solutions 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
Prepared by Tianxu Lan
Please send corrections, if any, to 1155184513@link.cuhk.edu.hk

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Remark. If you do not have the background in elementary probability theory, then the first three chapters of the textbook *Probability, Statistics, and Stochastic Processes* by Mikael Andersson and Peter Olofsson may be a good reference for you.

1.

Let X be a discrete random variable that has a binomial distribution with parameters n and p, written as $X \sim \text{Binomial}(n, p)$. Its probability mass function is given by

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots$$

where $p \in (0, 1)$ is some constant. Compute the following values:

a) $E[X]$, $E[X^2]$ and hence $\text{Var}[X]$.

b) $M_X(t) := E[\exp(tX)]$, where $t \in \mathbb{R}$.

c) The derivatives at $t = 0$:

$$\frac{d}{dt} M_X(t)\bigg|_{t=0} \quad \text{and} \quad \frac{d^2}{dt^2} M_X(t)\bigg|_{t=0}.$$

These values should agree with the values of $E[X]$ and $E[X^2]$ that you have obtained in part (a).

Hint 1: For a discrete random variable X, the expectation value of the random variable $g(X)$ is given by $E[g(X)] = \sum_x g(x) P(X = x)$. Here $g(X)$ is any function of X, for example, you may take $g(X) = X^2$.

Hint 2: You may find the binomial theorem useful:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Solution.
a) We perform straightforward computation following the definition of E:

$$E[X] = \sum_{k=0}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} \quad \text{(the } k=0 \text{ term is zero)}$$

$$= \sum_{k=1}^{\infty} n \binom{n-1}{k-1} p^k (1-p)^{n-k} = np \sum_{m=0}^{\infty} \binom{n-1}{m} p^m (1-p)^{(n-1)-m} = np[p+(1-p)]^{n-1} = np.$$

Similarly,

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{\infty} k^2 \binom{n}{k} p^k (1-p)^{n-k} \quad \text{(the } k=0 \text{ term is zero)}$$

$$= \sum_{k=1}^{\infty} kn \binom{n-1}{k-1} p^k (1-p)^{n-k} = np \sum_{m=0}^{\infty} (m+1) \binom{n-1}{m} p^{m+1} (1-p)^{(n-1)-(m+1)}.$$

The first sum can be found by the same method of obtaining $E[X]$ and the second sum can be found by the binomial theorem.

Thus, the variance is

$$\text{Var}[X] = E[X^2] - E[X]^2 = n(n-1)p^2 + np - (np)^2 = np(1-p).$$

b) This is even more straightforward by using the binomial theorem:

$$M_X(t) = E[\exp(tX)] = \sum_{k=0}^{\infty} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{\infty} \binom{n}{k} (pe^t)^k (1-p)^{n-k} = [pe^t+(1-p)]^n.$$

c) A direct computation shows that

$$M'(t) = n[pe^t + (1-p)]^{n-1} \cdot pe^t = np[pe^t + (1-p)]^{n-1} \cdot e^t.$$

$$M''(t) = n(n-1)p[pe^t + (1-p)]^{n-2} \cdot pe^t \cdot e^t + np[pe^t + (1-p)]^{n-1} \cdot e^t.$$

Hence, we have that

$$M'(0) = np \quad \text{and} \quad M''(0) = n(n-1)p^2 + np.$$

Remark. This $M_X(t)$ is called the moment generating function of X. For the reason why its derivatives are equal to $E[X]$ and $E[X^2]$, see Corollary 3.15 in the textbook.

2.

Let X be a continuous random variable that has a normal distribution with parameters μ and σ^2, written as $X \sim N(\mu, \sigma^2)$. Its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$, and $\sigma > 0$ are constants. Compute the following values:

a) $E[X]$, $E[X^2]$ and hence $\mathrm{Var}[X]$.

b) $M_X(t) := E[\exp(tX)]$, where $t \in \mathbb{R}$ is some constant.

c) The derivatives at $t = 0$:

$$\left.\frac{d}{dt} M_X(t)\right|_{t=0} \quad \text{and} \quad \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0}.$$

These values should agree with the values of $E[X]$ and $E[X^2]$ that you have obtained in part (a).

Hint 3: For a continuous random variable X, the expectation value of the random variable $g(X)$ is given by $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$. Here $g(X)$ is any function of X, for example, you may take $g(X) = X^2$.

Hint 4: You may find the following integral helpful:

$$\int_0^\infty e^{-z^2/2}\, dz = \frac{\sqrt{\pi}}{2}.$$

Solution.

a)

$$E[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-\frac{z^2}{2}} \sigma\, dz \quad (\text{change of variable } z = \frac{x-\mu}{\sigma})$$

$$= \mu \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz = \mu \quad (\text{the first integrand is odd}).$$

And

$$E[X^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\sigma z + \mu)^2 e^{-\frac{z^2}{2}} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sigma^2 z^2 + 2\mu\sigma z + \mu^2\right) e^{-\frac{z^2}{2}}\, dz = \sigma^2 \cdot \sqrt{2\pi} \cdot \frac{1}{2} + \mu^2.$$

Hence, $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \sigma^2$.

b)

$$M_X(t) = E[\exp(tX)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} e^{-\frac{z^2}{2}} \sigma \, dz \quad (\text{change of variable } z = \frac{x-\mu}{\sigma})$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}+\sigma t z} \, dz = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma t)^2}{2}} \, dz \quad (\text{completing the square}).$$

c)

$$M'(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot (\mu + \sigma^2 t) \quad \text{and} \quad M''(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \left(\mu + \sigma^2 t\right)^2 + e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \sigma^2.$$

Therefore,

$$M'(0) = \mu \quad \text{and} \quad M''(0) = \mu^2 + \sigma^2.$$

3.

Recall that (Corollary 3.8 in the textbook) if two random variables are independent, then they are uncorrelated, i.e. $\mathrm{Cov}(X, Y) = 0$. However, the converse is not true in general and this problem provides an example. Let X be a random variable with continuous uniform distribution on the interval $[-1, 1]$, i.e. its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

a Show that $\mathrm{Cov}(X, X^2) = 0$.

b Prove mathematically (not just argue by intuition) that X and X^2 are not independent. One way to do this is by showing that they do not satisfy the property:

$$P(X \in A, X^2 \in B) = P(X \in A) \cdot P(X^2 \in B)$$

for all $A, B \subseteq \mathbb{R}$. You may also use other equivalent definitions of independence.

Solution.

a) Since the integrands are odd, we have

$$\mathrm{Cov}(X, X^2) = E[(X - \mu_X)(X^2 - \mu_{X^2})] = E[X^3] - \mu_X \mu_{X^2} = 0$$

where $\mu_X = E[X]$ and $\mu_{X^2} = E[X^2]$.

b) First of all, we remark that it is obvious that X and X^2 cannot be independent of each other. The problem is how to prove it mathematically. Here we provide one method which is to explicitly construct some $A, B \subseteq \mathbb{R}$ such that

$$P(X \in A, X^2 \in B) \neq P(X \in A) \cdot P(X^2 \in B).$$

Let $0 < a < 1$ and denote $A = B = (-\infty, a]$. Then, on one hand, we have

$$P(X \in A, X^2 \in B) = P(X \leq a, X^2 \leq a) = P(X \leq a, -\sqrt{a} \leq X \leq \sqrt{a}) = P(-\sqrt{a} \leq X \leq a).$$

$$= \int_{-\sqrt{a}}^{a} f(x)\, dx = \frac{1}{2} \cdot (a + \sqrt{a}).$$

On the other hand, we have

$$P(X \in A) \cdot P(X^2 \in B) = P(X \leq a) \cdot P(-\sqrt{a} \leq X \leq a) = \left(\int_{-1}^{a} \frac{1}{2}\, dx \right) \cdot \left(\int_{-\sqrt{a}}^{\sqrt{a}} \frac{1}{2}\, dx \right) = (a+1) \cdot \left(\sqrt{a} \right).$$

Therefore, we have

$$P(X \in A, X^2 \in B) = \frac{1}{2}(a + \sqrt{a}) \neq (a+1) \cdot \frac{1}{2}\sqrt{a} = P(X \in A) \cdot P(X \in B)$$

since $0 < a < 1$.