Homework 4 Solutions 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
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Please submit your solutions on blackboard before
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1. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$.

(a) Let τ_1, τ_2 be two $\mathcal{F}$-stopping times. Prove that

$$\tau_1 \wedge \tau_2 := \min(\tau_1, \tau_2), \quad \tau_1 \vee \tau_2 := \max(\tau_1, \tau_2)$$

are both stopping times.

(b) Let τ be an $\mathcal{F}$-stopping time. Prove that $\tau + 1$ is also an $\mathcal{F}$-stopping time.

2. Let $X_0 = 0$, $X_n = \sum_{k=1}^{n} \xi_k$, where $(\xi_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables such that $P[\xi_k = \pm 1] = \frac{1}{2}$. Let M and N be two positive integers and define

$$\tau := \min\{n \geq 0 : X_n = -N \text{ or } X_n = M\}.$$

(a) Prove that τ is an $\mathcal{F}$-stopping time, where $\mathcal{F}$ is the natural filtration generated by X.

(b) Assume that $\tau < +\infty$ a.s., prove that $P[X_\tau \in \{-N, M\}] = 1$.

(c) Under the condition of (b), compute $E[X_\tau]$ and $P[X_\tau = -N]$.

Hint: Let X be a martingale and τ be a stopping time with respect to a filtration $\mathcal{F}$, and if $\tau < \infty$ and the process $(X_{\tau\wedge n})_{n\geq 0}$ is uniformly bounded, then $E[X_\tau] = E[X_0]$.