Homework 3

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
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Please submit your solutions on blackboard before
11:59 AM, Jan 27th 2025

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1.

(a) Let $X : \Omega \to \mathbb{R}$ be a random variable such that $X \equiv 0$, i.e. for any $\omega \in \Omega$ $X(\omega) = 0.$, Prove that $\sigma(X) = \{\emptyset, \Omega\}$.
(b) Let $G := \{\emptyset, \Omega\}$, and $X : \Omega \to \mathbb{R}$ be G-measurable. Prove that $X \equiv c$ for some constant $c \in \mathbb{R}$.

2.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be a filtration. Given an $\mathcal{F}$-predictable process $(H_n)_{n\geq 0}$, which is uniformly bounded, and an $\mathcal{F}$-martingale $(X_n)_{n\geq 0}$, we define a process $(V_n)_{n\geq 0}$ by

$$V_0 := 0, \quad V_n := \sum_{k=1}^{n} H_k(X_k - X_{k-1}).$$

Prove that $(V_n)_{n\geq 0}$ is still an $\mathcal{F}$-martingale.

3.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be a filtration. Given an $\mathcal{F}$-submartingale $(X_n)_{n\geq 0}$, we define

$$\Delta A_n := E[X_n|\mathcal{F}_{n-1}] - X_{n-1}, \quad \Delta M_n := X_n - E[X_n|\mathcal{F}_{n-1}], \quad \forall n \geq 1,$$

and

$$A_0 = M_0 = 0, \quad A_n := \sum_{k=1}^{n} \Delta A_k, \quad M_n := \sum_{k=1}^{n} \Delta M_k.$$

(a) Prove that $(M_n)_{n\geq 0}$ is an $\mathcal{F}$-martingale, and that $(A_n)_{n\geq 0}$ is an increasing $\mathcal{F}$-predictable process.

(b) Prove that $(X_n)_{n\geq 0}$ has the decomposition

$$X_n = X_0 + M_n + A_n, \quad \forall n \geq 0.$$

(c) Let $(A_n^1)_{n\geq 0}$ and $(A_n^2)_{n\geq 0}$ be two $\mathcal{F}$-predictable processes such that $A_0^1 = A_0^2 = 0$. Prove that if $(A_n^1 - A_n^2)_{n\geq 0}$ is an $\mathcal{F}$-martingale, then $A_n^1 = A_n^2$, a.s. for each $n \geq 1$.

(d) Deduce that the decomposition (1) is unique, i.e. if one has

$$X_n = X_0 + \tilde{M}_n + \tilde{A}_n, \quad \forall n \geq 0,$$

for some $\mathcal{F}$-martingale $(\tilde{M}_n)_{n\geq 0}$ and increasing $\mathcal{F}$-predictable process $(\tilde{A}_n)_{n\geq 0}$ such that $\tilde{M}_0 = \tilde{A}_0 = 0$, then $A_n = \tilde{A}_n$ and $M_n = \tilde{M}_n$, a.s. for each $n \geq 1$.