Homework 2 Solutions 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
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1.

Let X and Y be two random variables with the same discrete uniform distributions, i.e., their probability mass functions are given by

$$P(X = k) = P(Y = k) = \frac{1}{N}, \quad k \in \{1, 2, \ldots, N\}.$$

Suppose that X and Y are independent. Compute $E[X + Y|X]$.

Hint: You may want to recall the elementary formula

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Solution. We have

$$E[Y] = \sum_{k=1}^{N} k \cdot P(Y = k) = \frac{1}{N} \sum_{k=1}^{N} k = \frac{N+1}{2}.$$

Thus,

$$E[X + Y|X] = E[X|X] + E[Y|X] = X + E[Y] = X + \frac{N+1}{2}.$$

2.

Suppose that X and Y are independent random variables, both with the standard normal distribution, i.e., $X, Y \sim N(0, 1)$. For $\rho \in [-1, 1]$, define

$$Z := \sqrt{1 - \rho^2} X + \rho Y.$$

Show that $E[Z|Y] = \rho Y$.

Solution.

$$E[Z|Y] = E\left[\sqrt{1 - \rho^2} X + \rho Y|Y\right] = \sqrt{1 - \rho^2} E[X|Y] + \rho E[Y|Y] = \sqrt{1 - \rho^2} E[X] + \rho Y = \rho Y.$$

3.

Fix a probability space $(\Omega, \mathcal{F}, P)$. Let X and Y be two independent continuous random variables with joint density function $\rho_{X,Y}(x,y)$ and the marginal density functions are denoted by $\rho_X(x)$ and $\rho_Y(y)$. Let $g(X,Y)$ be a function with $E[|g(X,Y)|] < \infty$ and we define

$$f(y) := E[g(X,y)].$$

Show that

$$E[g(X,Y)|Y] = f(Y).$$

Hint: Use Fubini's theorem; also, the proof of Example 1.16 (ii) in the lecture notes may be helpful.

Solution. We shall show that $f(Y)$ satisfies all the criteria listed in the definition of conditional expectation.

(a) It is clear that $f(Y)$ is $\sigma(Y)$-measurable.

(b) $f(Y)$ is integrable:

$$E[|f(Y)|] = \int |f(y)|\rho_Y(y)dy \quad \text{by definition of expectation}$$

$$= \int \left(\int g(x,y)\rho_X(x)dx \right) \rho_Y(y)dy \quad \text{by definition of } f(y)$$

$$\leq \int \left(\int |g(x,y)|\rho_X(x)dx \right) \rho_Y(y)dy \quad \text{since } \int \cdots | \leq \int \cdots |$$

$$= \int \int |g(x,y)|\rho_X(x)\rho_Y(y)dxdy \quad \text{by Fubini's theorem}$$

$$= \int \int |g(x,y)|\rho_{X,Y}(x,y)dxdy \quad \text{by independence of } X \text{ and } Y$$

$$= E[|g(X,Y)|] \quad \text{by definition of expectation}$$

$$< \infty \quad \text{by the hypothesis}$$

(c) We need to establish that for all $\sigma(Y)$-measurable bounded random variable Z, we must have $E[g(X,Y)Z] = E[f(Y)Z]$.

Indeed, since Z is $\sigma(Y)$-measurable, there exists some measurable function h such that $Z = h(Y)$. It then follows that

$$E[g(X,Y)Z] = E[g(X,Y)h(Y)] = \int \int g(x,y)h(y)\rho_{X,Y}(x,y)dxdy.$$

By independence,

$$= \int \left(\int g(x,y)\rho_X(x)dx \right) h(y)\rho_Y(y)dy = \int f(y)h(y)\rho_Y(y)dy = E[f(Y)h(Y)] = E[f(Y)Z].$$