Homework 1 Solutions 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
Prepared by Tianxu Lan
1155184513@link.cuhk.edu.hk
Please submit your solutions on blackboard before
11:59 AM, Jan 13th 2025

Remark. If you do not have the background in elementary probability theory, then the first three chapters of the textbook *Probability, Statistics, and Stochastic Processes* by Mikael Andersson and Peter Olofsson may be a good reference for you.

1. Binomial Distribution

Let X be a discrete random variable that has a binomial distribution with parameters n and p, written as $X \sim \text{Binomial}(n, p)$. Its probability mass function is given by

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots$$

where $p \in (0, 1)$ is some constant. Compute the following values:

(a) $E[X]$, $E[X^2]$ and hence $\text{Var}[X]$.

(b) $M_X(t) := E[\exp(tX)]$, where $t \in \mathbb{R}$.

(c) The derivatives at $t = 0$:

$$\frac{d}{dt} M_X(t)\bigg|_{t=0} \quad \text{and} \quad \frac{d^2}{dt^2} M_X(t)\bigg|_{t=0}.$$

These values should agree with the values of $E[X]$ and $E[X^2]$ that you have obtained in part (a).

Hint 1: For a discrete random variable X, the expectation value of the random variable $g(X)$ is given by $\sum_x g(x)P(X = x)$. Here $g(X)$ is any function of X, for example, you may take $g(X) = X^2$.

Hint 2: You may find the binomial theorem useful:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. Normal Distribution

Let X be a continuous random variable that has a normal distribution with parameters μ and σ^2, written as $X \sim N(\mu, \sigma^2)$. Its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

Compute the following values:

(a) $E[X]$, $E[X^2]$ and hence $\text{Var}[X]$.

(b) $M_X(t) := E[\exp(tX)]$, where $t \in \mathbb{R}$.

(c) The derivatives at $t = 0$:

$$\frac{d}{dt} M_X(t) \bigg|_{t=0} \quad \text{and} \quad \frac{d^2}{dt^2} M_X(t) \bigg|_{t=0}.$$

These values should agree with the values of $E[X]$ and $E[X^2]$ obtained in part (a).

Hint 3: For a continuous random variable X, the expectation value of $g(X)$ is given by $\int_{-\infty}^{\infty} g(x) f(x) dx$.

Hint 4: You may find the following integral helpful:

$$\int_0^\infty e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}}.$$

3. Covariance and Independence

Recall that (Corollary 3.8 in the textbook) if two random variables are independent, then they are uncorrelated, i.e., $\text{Cov}(X, Y) = 0$. However, the converse is not true in general and this problem provides an example. Let X be a random variable with a continuous uniform distribution on the interval $[-1, 1]$, i.e., its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $\text{Cov}(X, X^2) = 0$.

(b) Prove mathematically (not just argue by intuition) that X and X^2 are not independent.