Homework 5 Solutions 2024-2025

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Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
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1.

Let $(\xi_k)_{k\geq 1}$ be a sequence of independent and identically distributed random variables with standard Gaussian distribution, i.e. $\xi_k \sim N(0,1)$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0, \quad X_n := \sum_{k=1}^{n} \frac{1}{k}\xi_k, \text{ for all } n \geq 1.$$

(a) Prove that X is a martingale.

Proof. We only check the martingale property (but you need to check other properties). Indeed,

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E\left[\frac{1}{n+1}\xi_{n+1}|\mathcal{F}_n\right] = X_n + \frac{1}{n+1}E[\xi_{n+1}] = X_n$$

by measurability of X_n with respect to $\mathcal{F}_n$ and independence of ξ_{n+1} with respect to $\mathcal{F}_n$.

(b) Prove that $\sup_{n\in\mathbb{N}} E[|X_n|^2] < \infty$.

Hint: You may use the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ without proof.

Proof. Note the elementary identity:

$$|X_n|^2 = \left(\sum_{k=1}^{n} \frac{1}{k}\xi_k\right)^2 = \sum_{k=1}^{n} \frac{1}{k^2}\xi_k^2 + 2\sum_{i\neq j} \frac{1}{i}\xi_i \cdot \frac{1}{j}\xi_j.$$

Taking the expectation, we have

$$E[|X_n|^2] = \sum_{k=1}^{n} \frac{1}{k^2}E[\xi_k^2] + 2\sum_{i\neq j} \frac{1}{i}\frac{1}{j}E[\xi_i\xi_j].$$

Since $E[\xi_k^2] = \mathrm{Var}[\xi_k] + E[\xi_k]^2 = 1$ and $\{\xi_k\}_{k\geq 1}$ are independent, it follows that

$$E[|X_n|^2] = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Hence, we have

$$\sup_n E[|X_n|^2] \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

(c)

By the convergence theorem of the martingale (Theorem 2.4), we know that $X_n \to X_\infty$ a.s. and in L^2 for some random variable X_∞ as $n \to \infty$.

(i)

Compute the characteristic function ψ_n of X_n, where ψ_n is defined as

$$\psi_n(\theta) := E[e^{i\theta X_n}], \quad \theta \in \mathbb{R}.$$

(ii)

Compute

$$\psi(\theta) := \lim_{n\to\infty} \psi_n(\theta), \quad \theta \in \mathbb{R}.$$

(iii)

Identify the distribution of X_∞. Hint: ψ is the characteristic function of X_∞ and the distribution of a random variable is uniquely determined by its characteristic function.

Proof. First note that

$$\xi_k \sim N(0,1) \implies \frac{1}{k}\xi_k \sim N(0, \frac{1}{k^2}) \implies X_n = \sum_{k=1}^{n} \frac{1}{k}\xi_k \sim N\left(0, \sum_{k=1}^{n} \frac{1}{k^2}\right).$$

Knowing the distribution of X_n, we compute ψ_n directly by evaluating the integral. Denote $\sigma_n^2 := \sum_{k=1}^{n} \frac{1}{k^2}$. We have, by definition,

$$\psi_n(\theta) = E[e^{i\theta X_n}] = \int e^{i\theta x} \cdot \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x^2}{2\sigma_n^2}}\, dx.$$

Now, since

$$\frac{d}{d\theta} e^{i\theta X_n} = (iX_n)e^{i\theta X_n},$$

and X_n is integrable, we can differentiate under the integral. Hence, we have

$$\psi_n'(\theta) = \int (ix)e^{i\theta x} \cdot \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x^2}{2\sigma_n^2}}\, dx = -\theta\sigma_n^2\psi_n(\theta).$$

Solving this differential equation with the initial condition $\psi_n(0) = 1$, we obtain

$$\psi_n(\theta) = e^{-\frac{\sigma_n^2}{2}\theta^2}, \quad \text{where } \sigma_n^2 := \sum_{k=1}^{n} \frac{1}{k^2}.$$

The limit is

$$\psi(\theta) = \lim_{n\to\infty} \psi_n(\theta) = e^{-\frac{\sigma^2}{2}\theta^2}, \quad \text{where } \sigma^2 := \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Thus, $X_\infty \sim N(0, \sigma^2) = N(0, \frac{\pi^2}{6})$.

2.

Let $(\xi_k)_{k\geq 1}$ be a sequence of independent and identically distributed random variables such that $P[\xi_k = \pm 1] = \frac{1}{2}$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0, \quad X_n := \sum_{k=1}^{n} 2^{k-1}\xi_k 1_{\{k\leq\tau\}}, \text{ where } \tau := \inf\{k \in \mathbb{N} : \xi_k = 1\}.$$

(a) Prove that X is a martingale.

Proof. We only check the martingale property. Indeed, we have

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[2^n\xi_{n+1}1_{\{n+1\leq\tau\}}|\mathcal{F}_n] = X_n + 2^n E[\xi_{n+1}1_{\{n+1\leq\tau\}}] = X_n.$$

(b) Compute $P[\tau > n]$ and deduce that $P[\tau < +\infty] = 1$.

Hint: $\{\tau > n\} = \{\xi_1 = \cdots = \xi_n = -1\}$.

Proof.

$$P[\tau > n] = P(\{\xi_1 = \xi_2 = \cdots = \xi_n = -1\}) = P(\xi_1 = -1) \cdot P(\xi_2 = -1) \cdots P(\xi_n = -1) = \left(\frac{1}{2}\right)^n.$$

Thus,

$$P(\tau = \infty) = P\left(\bigcup_{n=1}^{\infty} \{\tau > n\}\right) = \lim_{n\to\infty} P[\tau > n] = 0 \implies P[\tau < \infty] = 1.$$

(c) Prove that $X_\tau = 1$ a.s.

Remark: It may be worth noting that in this case, we have $1 = E[X_\tau] \neq E[X_0] = 0$.

Proof.

$$X_\tau = \sum_{k=1}^{\tau} 2^{k-1}\xi_k 1_{\{k\le\tau\}} = \sum_{k=1}^{\tau} 2^{k-1}\xi_k = -\sum_{k=1}^{\tau} 2^{k-1} + 2^{\tau-1} = -(2^{\tau-1} - 1) + 2^{\tau-1} = 1$$

which holds for all $\omega \in \Omega$ with $\tau(\omega) < \infty$, i.e., $X_\tau = 1$ a.s.

(d) Compute $E[|X_n|]$ and prove that $\sup_{n\in\mathbb{N}} E[|X_n|] < \infty$, and $\lim_{n\to\infty} X_n = X_\tau$ a.s.

Hint: $E[|X_n|] = E[|X_n| 1_{\{\tau>n\}}] + E[|X_\tau| 1_{\{\tau\le n\}}]$.

Proof.

$$E[|X_n| 1_{\{\tau>n\}}] = E\left[\sum_{k=1}^{n} 2^{k-1}(-1) 1_{\{\tau>n\}}\right] = \sum_{k=1}^{n} 2^{k-1} \cdot P(\tau > n) = (2^n - 1) \cdot \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

and

$$E[|X_\tau| 1_{\{\tau\le n\}}] = E\left[\sum_{k=1}^{\tau} 2^{k-1}\xi_k 1_{\{\tau\le n\}}\right] = E\left[-\sum_{k=1}^{\tau} 2^{k-1}(-1) + 2^{\tau-1}(1)\right] 1_{\{\tau\le n\}} = P(\tau \le n) = 1 - \frac{1}{2^n}.$$

Hence,

$$E[|X_n|] = 2 - \frac{1}{2^{n-1}} \quad \text{and} \quad \sup_n E[|X_n|] < \infty.$$

For the pointwise limit, note that for $\omega \in \Omega$ with $\tau(\omega) < \infty$, we have $|X_n - X_\tau| = 0$ for $n \ge \tau(\omega)$. The result follows as $P(\tau < \infty) = 1$.

You may also compute the limit directly:

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} X_n 1_{\{\tau>n\}} + X_\tau 1_{\{\tau\le n\}} = X_\tau \text{ a.s.}$$