MMAT 5010 Linear Analysis

Suggested Solution of Test

1. (10 points): Assume that $\mathbb{C}^2$ is equipped with the usual norm, i.e., $\|z\| := \sqrt{|z_1|^2 + |z_2|^2}$ for $z = (z_1, z_2) \in \mathbb{C}^2$. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear map given by the following matrix, that is $T(z) = Az$ for $z \in \mathbb{C}^2$.

$$A := \begin{bmatrix} -3 & 0 \\ 0 & 1+i \end{bmatrix}.$$

Find $\|T\|$.

Solution. For all $z \in \mathbb{C}^2$ with $\|z\| \leq 1$,

$$\|Az\| = \sqrt{|-3z_1|^2 + |(1+i)z_2|^2} = \sqrt{9|z_1|^2 + 2|z_2|^2} \leq 3\sqrt{|z_1|^2 + |z_2|^2} = 3.$$

Thus $\|T\| \leq 3$. On the other hand, if we take $e_1 = (1, 0) \in \mathbb{C}^2$, then $\|e_1\| = 1$ and

$$\|Ae_1\| = \sqrt{|-3|^2 + |0|^2} = 3.$$

Consequently $\|T\| = 3$. ◀

2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Define the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $X \oplus Y$ by

$$\|(x, y)\|_1 := \|x\|_X + \|y\|_Y \quad \text{and} \quad \|(x, y)\|_2 := \sqrt{\|x\|_X^2 + \|y\|_Y^2}$$

for $(x, y) \in X \oplus Y$.

(a) (10 points): Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

(b) (10 points): Show that if X and Y both are Banach spaces, then so is $X \oplus Y$ under the norm $\|\cdot\|_1$.

Solution. (a) For each $(x, y) \in X \oplus Y$, we have

$$\|(x, y)\|_1 = \|x\|_X + \|y\|_Y \leq 2 \cdot \sqrt{\|x\|_X^2 + \|y\|_Y^2} = 2\|(x, y)\|_2$$

and

$$\|(x, y)\|_2 \leq \sqrt{\|x\|_X^2 + 2\|x\|_X\|y\|_Y + \|y\|_Y^2} = \sqrt{(\|x\|_X + \|y\|_Y)^2} = \|(x, y)\|_1.$$

Hence, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on $X \oplus Y$.

(b) Suppose X and Y both are Banach spaces. Let $((x_n, y_n))$ be a Cauchy sequence in $X \oplus Y$. Since

$$\|x_n - x_m\|_X, \|y_n - y_m\|_Y \leq \|x_n - x_m\|_X + \|y_n - y_m\|_Y = \|(x_n, y_n) - (x_m, y_m)\|_1,$$

(x_n) and (y_n) are Cauchy sequences in X and Y, respectively. As X, Y are Banach spaces, there is $x \in X$ and $y \in Y$ such that

$$\|x_n - x\|_X, \|y_n - y\|_Y \to 0 \quad \text{as } n \to \infty.$$

Now, $((x_n, y_n))$ converges to (x, y) in $X \oplus Y$ because

$$\|(x_n, y_n) - (x, y)\|_1 = \|x_n - x\|_X + \|y_n - y\|_Y \to 0.$$

Therefore $(X \oplus Y, \|\cdot\|_1)$ is also a Banach space.

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3. Let Y be a proper subspace of a normed space X. Let $\pi : X \to X/Y$ be the natural projection. Define
$$q(\pi(x)) := \inf\{\|x + y\| : y \in Y\}$$
for $x \in X$.

   (a) (10 points): Show that $q : X/Y \to [0, \infty)$ is a well defined function, that is $\inf\{\|x + y\| : y \in Y\} = \inf\{\|x' + y\| : y \in Y\}$ whenever $\pi(x) = \pi(x')$.

   (b) (10 points): Show that if Y is closed, then q is a norm function on X/Y. In this case, show that $\|\pi\| = 1$.

   (Hint: use the Riesz' Lemma: for any $0 < \theta < 1$, there is $x_0 \in X$ with $\|x_0\| = 1$ such that $\|x_0 - y\| \geq \theta$ for all $y \in Y$.)

Solution. (a) Suppose $\pi(x) = \pi(x')$. Then $x - x' \in Y$, that is $x = x' + y'$ for some $y' \in Y$. Now, for any $y \in Y$,

$$\|x + y\| = \|x' + y' + y\| \geq \inf\{\|x' + z\| : z \in Y\}$$

since $y' + y \in Y$.

As $y \in Y$ is arbitrary, we have $\inf\{\|x + y\| : y \in Y\} \geq \inf\{\|x' + y\| : y \in Y\}$.
Similarly, we can show that $\inf\{\|x + y\| : y \in Y\} \leq \inf\{\|x' + y\| : y \in Y\}$.

(b) Clearly $q(\pi(x)) \geq 0$ for any $x \in X$.

   (i) Since Y is closed, one have $q(\pi(x)) = 0$ if and only if $x \in Y$, that is $\pi(x)$ is the zero vector in X/Y.

   (ii) Since Y is a closed subspace, we have for $\alpha \in \mathbb{K}$,

$$q(\alpha\pi(x)) = q(\pi(\alpha x)) = \inf\{\|\alpha x + y\| : y \in Y\} = |\alpha| \inf\{\|x + y\| : y \in Y\} = |\alpha| q(\pi(x)).$$

   (iii) For any $x_1, x_2 \in X$, $y_1, y_2 \in Y$,

$$\|x_1 + y_1\| + \|x_2 + y_2\| \geq \|x_1 + x_2 + y_1 + y_2\| \geq q(\pi(x_1 + x_2))$$

because $y_1 + y_2 \in Y$. Taking infimum over y_1, y_2 and since $\pi(x_1 + x_2) = \pi(x_1) + \pi(x_2)$, we have $q(\pi(x_1) + \pi(x_2)) \leq q(\pi(x_1)) + q(\pi(x_2))$.

Therefore, q is a norm function on X/Y.

For any $x \in X$, we have

$$q(\pi(x)) = \inf\{\|x+y\| : y \in Y\} \leq \|x\|$$

since $0 \in Y$. Thus $\|\pi\| \leq 1$.

By the Riesz' Lemma, for any $0 < \theta < 1$, there is $x_0 \in X$ with $\|x_0\| = 1$ such that $\|x_0 - y\| \geq \theta$ for all $y \in Y$. In particular, $q(\pi(x_0)) \geq \theta$. Hence $\|\pi\| \geq \theta$ for any $\theta \in (0, 1)$. Consequently $\|\pi\| = 1$.

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