MMAT 5010 Linear Analysis

Suggested Solution of Homework 9

1. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Show that the inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ is continuous, that is, whenever the sequences $x_n \to x$ and $y_n \to y$ in X, we have $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

   From this show that if A is a subset of X, then $A^\perp := \{x \in X : x \perp y, \text{ for all } y \in A\}$ is a closed subset of X.

   Solution. By the defining properties of inner product and Cauchy-Schwarz ineqaulity, we have

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\ &\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\|. \end{aligned}$$

   If $x_n \to x$ and $y_n \to y$ in X, then $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$ and $\|y_n\| \to \|y\|$, forcing $|\langle x_n, y_n \rangle - \langle x, y \rangle| \to 0$. Therefore, the inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ is continuous.

   Suppose (x_n) is a sequence in $A^\perp$ that converges to x in X. Then, for all $y \in A$, we have $x_n \perp y$, that is $\langle x_n, y \rangle = 0$ for all $n \in \mathbb{N}$. By the continuity of $\langle \cdot, \cdot \rangle$, we have $\langle x, y \rangle = 0$, that is $x \in A^\perp$. Therefore $A^\perp$ is closed. ◀

2. Let $(X, \langle \cdot, \cdot \rangle_X)$ and $(Y, \langle \cdot, \cdot \rangle_Y)$ be Hilbert spaces. For $(x_1, y_1), (x_2, y_2) \in X \times Y$, put

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y} := \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y.$$

   Show that $\langle \cdot, \cdot \rangle_{X \times Y}$ is an inner product on the direct sum $X \times Y$ and it is a Hilbert space under this inner product.

   Solution. For $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, and $\alpha, \beta \in \mathbb{C}$,

   (i) $\langle (x_1, y_1), (x_1, y_1) \rangle_{X \times Y} = \langle x_1, x_1 \rangle_X + \langle y_1, y_1 \rangle_Y \geq 0$, and it is 0 iff $\langle x_1, x_1 \rangle_X = \langle y_1, y_1 \rangle_Y = 0$ iff $x_1 = 0_X$ and $y_1 = 0_Y$;

   (ii) $\overline{\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y}} = \overline{\langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y} = \overline{\langle x_1, x_2 \rangle_X} + \overline{\langle y_1, y_2 \rangle_Y}$
   $= \langle x_2, x_1 \rangle_X + \langle y_2, y_1 \rangle_Y = \langle (x_2, y_2), (x_1, y_1) \rangle_{X \times Y}$;

   (iii) $\langle \alpha(x_1, y_1) + \beta(x_2, y_2), (x_3, y_3) \rangle_{X \times Y} = \langle (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2), (x_3, y_3) \rangle_{X \times Y}$
   $= \langle \alpha x_1 + \beta x_2, x_3 \rangle_X + \langle \alpha y_1 + \beta y_2, y_3 \rangle_Y = \alpha \langle x_1, x_3 \rangle_X + \beta \langle x_2, x_3 \rangle_X + \alpha \langle y_1, y_3 \rangle_Y + \beta \langle y_2, y_3 \rangle_Y = \alpha \langle (x_1, y_1), (x_3, y_3) \rangle_{X \times Y} + \beta \langle (x_2, y_2), (x_3, y_3) \rangle_{X \times Y}$.

   Hence, $\langle \cdot, \cdot \rangle_{X \times Y}$ is an inner product on the $X \times Y$.

   To see that $(X \times Y, \langle \cdot, \cdot \rangle_{X \times Y})$ is complete, let (x_n, y_n) be a Cauchy sequence in $X \times Y$ under the norm

$$\|(x, y)\|_{X \times Y} := \sqrt{\langle (x, y), (x, y) \rangle_{X \times Y}} = \sqrt{\langle x, x \rangle_X + \langle y, y \rangle_Y} = \sqrt{\|x\|_X^2 + \|y\|_Y^2},$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norm on X and Y induced by their respective inner products. Then (x_n) is a Cauchy sequence in $(X, \|\cdot\|_X)$ and (y_n) is a Cauchy sequence in $(Y, \|\cdot\|_Y)$, since

$$\|x_n - x_m\|_X,\ \|y_n - y_m\|_Y \leq \|(x_n, y_n) - (x_m, y_m)\|_{X\times Y}.$$

Since X and Y are Hilbert spaces, there are $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|y_n - y\|_Y \to 0$. Now (x_n, y_n) converges to (x, y) in $(X \times Y, \|\cdot\|_{X\times Y})$ because

$$\|(x_n, y_n) - (x, y)\|_{X\times Y} = \sqrt{\|x_n - x\|_X^2 + \|y_n - y\|_Y^2} \to 0 \quad \text{as } n \to \infty.$$

Therefore $(X \times Y, \langle \cdot, \cdot \rangle_{X\times Y})$ is a Hilbert space.

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3. Let X be a Hilbert space and let $T, S \in B(X)$. Show that

   (a) $(TS)^* = S^*T^*$.

   (b) if T is invertible, that is $T^{-1} \in L(X)$ exists, then $(T^{-1})^* = (T^*)^{-1}$.

Solution. (a) By Proposition 8.3, the adjoint operators $T^*, S^* \in B(X)$ satisfy

$$(TSx, y) = (Sx, T^*y) = (x, S^*T^*y) \quad \text{for any } x, y \in X. \tag{$*$}$$

Moreover, $S^*T^* \in B(X)$ since $\|S^*T^*x\| \leq \|S^*\|\|T^*x\| \leq \|S^*\|\|T^*\|\|x\|$ for any $x \in X$. By Proposition 8.3, $(TS)^*$ is the unique element in $B(X)$ satisfying $(*)$. Therefore $(TS)^* = S^*T^*$.

(b) By Corollary 5.4, $T^{-1} \in B(X)$. The identity map $I : X \to X$ clearly satisfies $I^* = I$. By (a),

$$T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I, \quad \text{and} \quad (T^{-1})^*T^* = (TT^{-1})^* = I^* = I.$$

Therefore, T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

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