MMAT 5010 Linear Analysis

Suggested Solution of Homework 8

1. Suppose that the finite sequences space c_{00} is endowed with the sup-norm. For each $n = 1, 2, \ldots$, define $T_n : c_{00} \to c_{00}$ by

$$T_n(x)(k) := \begin{cases} kx(k) & \text{as} \quad 1 \le k \le n \\ 0 & \text{as} \quad k > n, \end{cases}$$

for $x \in c_{00}$ and $k = 1, 2, \ldots$.

   (a) Show that each T_n is bounded.

   (b) Show that $T(x) := \lim_n T_n(x)$ exists for all $x \in c_{00}$.

   (c) Show that T is unbounded.

Solution. (a) For any $x \in c_{00}$, $\sup_k |T_n(x)(k)| \le n \sup_k |x(k)|$, and hence $\|T_n(x)\|_\infty \le \|x\|_\infty$. So T_n is bounded with $\|T_n\| \le n$.

(b) Let $x \in c_{00}$. We will show that $\lim_n T_n(x) = y$, where y is the sequence in c_{00} defined by $y(k) := kx(k)$ for each $k \in \mathbb{N}$. Since $x \in c_{00}$, there is $N \in \mathbb{N}$ such that $x(k) = 0$ for all $k > N$. Then, if $n \ge N$, we have

$$\|T_n(x) - y\|_\infty = \sup_k |T_n(x)(k) - kx(k)| = \sup_{k>n} |0 - kx(k)| = 0.$$

Therefore $T(x) := \lim_n T_n(x) = y$.

(c) From (b), we see that

$$T(x)(k) = kx(k) \quad \text{for } k = 1, 2, \ldots.$$

For $n \in \mathbb{N}$, let $e_n \in c_{00}$ be defined by $e_n(k) = 1$ if $k = n$ and 0 otherwise. Then $\|e_n\|_\infty = 1$ while $\|Te_n\|_\infty = n$. So $\|T\| = \infty$ and T is unbounded.

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