MMAT 5010 Linear Analysis

Suggested Solution of Homework 7

1. Let X and Y be non-zero normed spaces. Let $B(X, Y)$ be the space of all bounded linear maps from X to Y.

   (a) Show that for any non-zero elements $x_0 \in X$, there is $f \in X^*$ such that $f(x_0) = 1$.

   (b) Show that for any pair of elements $x_0 \in X$ with $x_0 \neq 0$ and $y_0 \in Y$, there is a bounded linear map T from X to Y such that $T(x_0) = y_0$.

   (c) Let $x_1, x_2 \in X$. Show that if $x_1 \neq x_2$, then there is $T \in B(X, Y)$ such that $T(x_1) \neq T(x_2)$.

   (d) Show that $B(X, Y) \neq \{0\}$.

   Solution. (a) Let $Z = \mathbb{K}x_0$. Define $f_0 : Z \to \mathbb{K}$ by $f_0(\alpha x_0) = \alpha$ for $\alpha \in \mathbb{K}$. Then $f_0 \in Z^*$ ($\|f_0\| = \|x_0\|^{-1}$). By the Hahn-Banach Theorem, there exists a linear extension $f \in X^*$ of f_0 such that $\|f\| = \|f_0\|$. In particular, $f(x_0) = f_0(x_0) = 1$.

   (b) Let f be the bounded linear functional obtained in (a). Define the map $T : X \to Y$ by $T(x) = f(x)y_0$. Clearly $T(x_0) = f(x_0)y_0 = y_0$. Moreover it is easy to check that T is linear and $\|T(x)\| = |f(x)|\|y_0\| \leq \|f\|\|x\|\|y_0\| = \|f\|\|y_0\|\|x\|$. So $T \in B(X, Y)$.

   (c) Fix $y_0 \in Y \backslash \{0\}$. By (b), there is $T \in B(X, Y)$ such that $T(x_1 - x_2) = y_0$, that is, $T(x_1) - T(x_2) = y_0 \neq 0$.

   (d) It follows immediately from (c).

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2. Let X be a normed space. Show that for any element $x_0 \in X$, we have $\|x_0\| = \sup\{|f(x_0)| : f \in X^*; \|f\| \leq 1\}$.

   Solution. For any $f \in X^*$ with $\|f\| \leq 1$, we have $|f(x_0)| \leq \|f\|\|x_0\| \leq \|x_0\|$. Hence $\|x_0\| \geq \sup\{|f(x_0)| : f \in X^*; \|f\| \leq 1\}$.

   We may assume that $x_0 \neq 0$. Let $Y = \mathbb{K}x_0$. Define $f_0 : Y \to \mathbb{K}$ by $f_0(\alpha x_0) = \alpha\|x_0\|$ for $\alpha \in \mathbb{K}$. Then $f_0 \in Y^*$ with $\|f_0\| = 1$. By the Hahn-Banach Theorem, there exists a linear extension $f \in X^*$ of f_0 such that $\|f\| = \|f_0\| = 1$. In particular, $f(x_0) = f_0(x_0) = \|x_0\|$. Hence $\|x_0\| = \sup\{|f(x_0)| : f \in X^*; \|f\| \leq 1\}$. ◀