MMAT 5010 Linear Analysis

Suggested Solution of Homework 5

1. Let c_{00} be the finite sequence space. Assume that c_{00} is endowed with the $\|\cdot\|_\infty$-norm. Define a linear map $T : c_{00} \to c_{00}$ by $Tx(k) := \frac{1}{k}x(k)$ for $x \in c_{00}$ and $k = 1, 2, \ldots$.

   (a) Find $\|T\|$.

   (b) Show that T is a linear isomorphism, that is T is linear bijection, and the inverse T^{-1} is unbounded.

Solution. (a) For any $x \in c_{00}$, $\sup_k |Tx(k)| = \sup_k |\frac{1}{k}x(k)| \leq \sup_k |x(k)|$, and hence $\|Tx\|_\infty \leq \|x\|$. So $\|T\| \leq 1$. On the other hand, if $x = (1, 0, 0, \ldots)$, then $Tx = x$ and so $\|T\| \geq 1$. Hence $\|T\| = 1$.

(b) Define the linear map $S : c_{00} \to c_{00}$ by $Sx(k) := kx(k)$ for $x \in c_{00}$ and $k = 1, 2, \ldots$. It is clear that

$$STx = x \quad \text{and} \quad TSx = x \quad \text{for any } x \in c_{00}.$$

Thus T is a linear bijection and $T^{-1} = S$ is also a linear map.

However, $\|T^{-1}\|$ is unbounded. For $n \in \mathbb{N}$, let $e_n \in c_{00}$ be defined by $e_n(k) = 1$ if $k = n$ and 0 otherwise. Then $\|e_n\|_\infty = 1$ while $\|T^{-1}e_n\|_\infty = n$. So $\|T^{-1}\| = \infty$.

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2. Let $(X, \|\cdot\|)$ be a normed space. Assume that $T : X \to X$ is a linear isomorphism.

   (a) For each $x \in X$, put $\|x\|_0 := \|Tx\|$. Show that $\|\cdot\|_0$ is a norm on X.

   (b) Show that the inverse T^{-1} is bounded if and only if there is $c > 0$ such that $c\|x\| \leq \|Tx\|$ for all $x \in X$.

   (c) The norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent if and only if T and T^{-1} both are bounded.

Solution. (a) Clearly $\|x\|_0 = \|Tx\| \geq 0$ for any $x \in X$.

(i) If $\|x\|_0 = \|Tx\| = 0$, then $Tx = 0$ and hence $x = 0$ since $\|\cdot\|$ is a norm and T is a linear isomorphism.

(ii) For any $\alpha \in \mathbb{K}$ and $x \in X$, $\|\alpha x\|_0 = \|T(\alpha x)\| = \|\alpha Tx\| = |\alpha|\|Tx\| = |\alpha|\|x\|_0$ since $\|\cdot\|$ is a norm and T is linear.

(iii) For any $x, y \in X$, $\|x+y\|_0 = \|T(x+y)\| = \|Tx+Ty\| \leq \|Tx\|+\|Ty\| = \|x\|_0 + \|y\|_0$ since $\|\cdot\|$ is a norm and T is linear.

Hence $\|\cdot\|_0$ is a norm on X.

(b) Suppose there is $c > 0$ such that $c\|x\| \leq \|Tx\|$ for all $x \in X$. Then, for any $x \in X$, we have

$$\|T^{-1}x\| \leq \frac{1}{c}\|T(T^{-1}x)\| = \frac{1}{c}\|x\|.$$

Hence T^{-1} is bounded with $\|T^{-1}\| \leq \frac{1}{c}$.

Suppose T^{-1} is bounded. Then, for any $x \in X$,

$$\|x\| = \|T^{-1}(Tx)\| \leq \|T^{-1}\|\|Tx\|,$$

and so $\frac{1}{c}\|x\| \leq \|Tx\|$, where $c := \|T^{-1}\| + 1 > 0$.

(c) Suppose $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Then there are $c_1, c_2 > 0$ such that

$$c_1\|x\| \leq \|x\|_0 = \|Tx\| \leq c_2\|x\| \quad \text{for any } x \in X.$$

The first inequality shows that T^{-1} is bounded by (b), while the second one shows that T is bounded.

Suppose T and T^{-1} both are bounded. By (b) and the definition of $\|T\|$, there are $c_1, c_2 > 0$ such that

$$c_1\|x\| \leq \|x\|_0 = \|Tx\| \leq c_2\|x\| \quad \text{for any } x \in X.$$

Hence $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.

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NOTE: Question 1(b) above shows us that a linear isomorphism T is bounded but the inverse T^{-1} may not be bounded in general.