MMAT 5010 Linear Analysis

Suggested Solution of Homework 4

1. Suppose that the space $\mathbb{R}^2$ is endowed with the usual norm, that is $\|(x_1, x_2)\| := \sqrt{x_1^2 + x_2^2}$. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}.$$
Find $\|A\|$?

Solution. For any $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$,
$$A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -3x_2 \end{bmatrix},$$
so that
$$\|A\mathbf{x}\| = \sqrt{x_1^2 + 9x_2^2} \leq 3\sqrt{x_1^2 + x_2^2} = 3\|\mathbf{x}\|.$$
Thus $\|A\| \leq 3$. Next we will show that $\|A\| \geq 3$. Indeed, if $e_2 = (0,1)$, then $\|Ae_2\| = \sqrt{0^2 + 9 \cdot 1^2} = 3 = 3\|e_2\|$. Hence $\|A\| \geq 3$.

Therefore $\|A\| = 3$. ◀

2. Let X be a normed space and let $V : X \to X$ be an isometric isomorphism on X, that is V is a linear isomorphism and $\|Vx\| = \|x\|$ for all $x \in X$. Show that if $T \in L(X)$, then $\|VTV^{-1}\| = \|T\|$.

Solution. Note that V^{-1} is also an isometric isomorphism since $\|V^{-1}x\| = \|VV^{-1}x\| = \|x\|$ for all $x \in X$. So, for all $x \in X$,
$$\|VTV^{-1}x\| = \|TV^{-1}x\| \leq \|T\|\|V^{-1}x\| \leq \|T\|\|x\|.$$
Thus $\|VTV^{-1}\| \leq \|T\|$. Since V^{-1} is also an isometric isomorphism, we also have $\|T\| = \|V^{-1}VTV^{-1}V\| \leq \|VTV^{-1}\|$. Therefore $\|VTV^{-1}\| = \|T\|$. ◀

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. For each element $(x, y) \in X \oplus Y$, put
$$\|(x,y)\|_1 := \|x\|_X + \|y\|_Y.$$
Let $T \in L(X)$ and $S \in L(Y)$. Define $T \oplus S : X \oplus Y \to X \oplus Y$ by $(T \oplus S)(x, y) := (Tx, Sy)$. Show that $\|T \oplus S\| = \max(\|T\|, \|S\|)$.

Solution. For any $(x, y) \in X \oplus Y$,
$$\begin{aligned} \|(T \oplus S)(x,y)\|_1 = \|(Tx, Sy)\|_1 = \|Tx\|_X + \|Sy\|_Y &\leq \|T\|\|x\|_X + \|S\|\|y\|_Y \\ &\leq \max(\|T\|, \|S\|)(\|x\|_X + \|y\|_Y) = \max(\|T\|, \|S\|)\|(x,y)\|_1. \end{aligned}$$
So $\|T \oplus S\| \leq \max(\|T\|, \|S\|)$.

On the other hand, for any $x \in B_X$, we have $(x, 0) \in B_{X \oplus Y}$ since $\|(x,0)\|_1 = \|x\|_X + \|0\|_Y \leq 1$, and thus
$$\|Tx\|_X = \|(T \oplus S)(x, 0)\|_1 \leq \|T \oplus S\|.$$
Taking supremum over $x \in B_X$, we have $\|T\| \leq \|T \oplus S\|$. Similarly, $\|S\| \leq \|T \oplus S\|$. ◀