MMAT 5010 Linear Analysis

Suggested Solution of Homework 3

1. Let $\|\cdot\|$ and $\|\cdot\|'$ be the norm functions defined on a vector space X. Let D be a subset of X. Show that if $\|\cdot\| \sim \|\cdot\|'$, then D is a compact with respect to the norm $\|\cdot\|$ if and only if it is also compact with respect to the norm $\|\cdot\|'$.

   Solution. Since $\|\cdot\| \sim \|\cdot\|'$, there are positive numbers c_1, c_2 such that $c_1\|\cdot\| \leq \|\cdot\|' \leq c_2\|\cdot\|$ on X.

   Suppose D is compact with respect to the norm $\|\cdot\|$. Let (x_n) be a sequence in D. Then (x_n) has a subsequence (x_{n_k}) converging to $x \in D$ with respect to the norm $\|\cdot\|$, that is $\|x_{n_k} - x\| \to 0$ as $k \to \infty$. So $\|x_{n_k} - x\|' \to 0$ as $k \to \infty$ because $\|\cdot\|' \leq c_2\|\cdot\|$. Thus the subsequence (x_{n_k}) converges to $x \in D$ with respect to the norm $\|\cdot\|'$ also. Hence D is compact with respect to the norm $\|\cdot\|'$.

   The converse can be proved by the same argument. ◀

2. Show that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are not equivalent on the finite sequence space c_{00}.

   Solution. For each $n \in \mathbb{N}$, let $x_n \in c_{00}$ be defined by

   $$x_n(k) = \begin{cases} 1 & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

   Then $\|x_n\|_\infty = 1$ while $\|x_n\|_2 = \sqrt{n}$. So there is no constant $C > 0$ such that $\|x_n\|_\infty \geq C\|x_n\|_2$ for all $n \in \mathbb{N}$. In particular, $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are not equivalent on c_{00}. ◀

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. For each element $(x, y) \in X \oplus Y$, put

   $$\|(x,y)\|_\infty := \max(\|x\|_X, \|y\|_Y) \quad \text{and} \quad \|(x,y)\|_1 := \|x\|_X + \|y\|_Y.$$

   Show that these two norms are equivalent on $X \oplus Y$.

   Solution. For each $(x, y) \in X \oplus Y$, we have

   $$\|(x,y)\|_\infty = \max(\|x\|_X, \|y\|_Y) \leq \|x\|_X + \|y\|_Y = \|(x,y)\|_1,$$

   and

   $$\|(x,y)\|_1 = \|x\|_X + \|y\|_Y \leq 2 \cdot \max(\|x\|_X, \|y\|_Y) = 2\|(x,y)\|_\infty.$$

   Hence, $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent norms on $X \oplus Y$. ◀