MMAT 5010 Linear Analysis

Suggested Solution of Homework 1

1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Now for each element $(x, y) \in X \oplus Y$ (the direct sum of X and Y) we put $\|(x, y)\|_\infty := \max(\|x\|_X, \|y\|_Y)$. Show that $(X \oplus Y, \|\cdot\|_\infty)$ is a Banach space if and only if X and Y both are Banach spaces.

   Solution. ($\Longrightarrow$) Suppose X and Y both are Banach spaces. Let $((x_n, y_n))$ be a Cauchy sequence in $X \oplus Y$. Since $\|x_n - x_m\|_X, \|y_n - y_m\|_Y \leq \|(x_n, y_n) - (x_m, y_m)\|_\infty$, (x_n) and (y_n) are Cauchy sequences in X and Y, respectively. As X, Y are Banach spaces, there is $x \in X$ and $y \in Y$ such that

   $$\|x_n - x\|_X, \|y_n - y\|_Y \to 0 \quad \text{as } n \to \infty.$$

   Now, $((x_n, y_n))$ converges to (x, y) in $X \oplus Y$ because

   $$\|(x_n, y_n) - (x, y)\|_\infty = \|(x_n - x, y_n - y)\|_\infty = \max(\|x_n - x\|_X, \|y_n - y\|_Y) \to 0.$$

   Therefore $(X \oplus Y, \|\cdot\|_\infty)$ is also a Banach space.

   ($\Longleftarrow$) Suppose $(X \oplus Y, \|\cdot\|_\infty)$ is a Banach space. Let (x_n) be a Cauchy sequence in X. Then $((x_n, 0_Y))$ is a Cauchy sequence in $X \oplus Y$ because $\|(x_n, 0_Y) - (x_m, 0_Y)\|_\infty = \|x_n - x_m\|_X$. As $X \oplus Y$ is a Banach space, there is $(x, y) \in X \oplus Y$ such that $\lim\limits_{n\to\infty} \|(x_n, 0_Y) - (x, y)\|_\infty = 0$. Since

   $$\|x_n - x\|_X \leq \|(x_n, 0_Y) - (x, y)\|_\infty,$$

   we must have $\lim\limits_{n\to\infty} \|x_n - x\|_X = 0$. Therefore X is a Banach space. Similarly, Y is also a Banach space. ◀

2. Let (x_n) be a sequence in a normed space X.

   (a) Suppose that there is $0 < r < 1$ such that $\|x_n\| < r^n$ for all $n = 1, 2, \ldots$. Put $s_n := \sum_{k=1}^n x_k$. Show that if X is a Banach space, then $\sum_n x_n := \lim_n s_n$ exists in X.

   (b) Consider the finite sequence space $(c_{00}, \|\cdot\|_\infty)$. For each $n = 1, 2, \ldots$, let $x_n(k) = 1/2^n$ as $k = n$, otherwise, set $x_n(k) = 0$, i.e. $x_n := (0, \ldots, 0, 1/2^n, 0, \ldots) \in c_{00}$ at the n-th position is $1/2^n$. We keep the notation as Part (a). Show that $\lim_n s_n$ does not exist in c_{00}.

   Solution. (a) Note that, for $m > n$,

   $$\|s_m - s_n\| = \|\sum_{k=n+1}^{m} x_k\| \leq \sum_{k=n+1}^{m} \|x_k\| < \sum_{k=n+1}^{m} r^k \leq \frac{r^{n+1}}{1-r}.$$

   Since $\lim\limits_{n\to\infty} \frac{r^{n+1}}{1-r} = 0$, (s_n) is a Cauchy sequence in the Banach space X. Thus $\lim_n s_n$ exists in X.

(b) Suppose $s := \lim_n s_n$ exists in c_{00}. Then for $m \geq n$, $s_m(n) = \sum_{k=1}^{m} x_k(n) = \frac{1}{2^n}$, and hence

$$|s(n) - \frac{1}{2^n}| = |s(n) - s_m(n)| \leq \|s - s_m\|_\infty \to 0 \qquad \text{as } m \to \infty.$$

Thus $s(n) = \frac{1}{2^n}$ for $n \in \mathbb{N}$. Such s cannot belong to c_{00} because it is not a finite sequence. Therefore $\lim_n s_n$ does not exist in c_{00}.

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